SoSe 24 Algebraic Topology II
Exercise sheet 4 (due June 14)

Exercise 4.1.

(a) Let $i\colon A \hookrightarrow B$ and $j\colon A \hookrightarrow C$ be embeddings such that either of the following conditions hold:

(i) i and j are open embeddings;

(ii) A, B, and C have CW structures such that i and j are subcomplexes.

Let G be any topological group. Show that the canonical functor

$$\mathrm{Bun}_G(B \sqcup_A C) \to \mathrm{Bun}_G(B) \times^h_{\mathrm{Bun}_G(A)} \mathrm{Bun}_G(C)$$

is an equivalence.

Hint. By definition, $\mathrm{Bun}_G(X)$ is a full subcategory of the category $\mathrm{Top}_G(X)$ of right G-spaces over X. First prove the analogous statement for Top_G (which holds more generally whenever i and j are closed embeddings). It then remains to show that the G-space E over $B \sqcup_A C$ obtained by gluing principal G-bundles $P \to B$ and $Q \to C$ along an isomorphism $P_A \cong Q_A$ is locally trivial. This is obvious in case (i). In case (ii), use the fact that subcomplexes are neighborhood retracts to show that E is locally isomorphic to a gluing of trivial G-bundles over some open neighborhoods of B and C in $B \sqcup_A C$.

Remark. There is an analogous statement with pointed spaces and $\mathrm{Bun}_{G,*}$.

(b) Show that for any pointed CW complex X, $[\Sigma X, BG]_* \cong [X, G]_*$. In particular, $\pi_{n+1}(BG) \cong \pi_n(G)$ for all $n \geq 0$.

Hint. Use the pointed version of (a) to compute $\pi_0\mathrm{Bun}_{G,*}(\Sigma X)$ explicitly.

Exercise 4.2.

(a) Let $X, Y \in \mathsf{Top}_*$ with X well-pointed and Y path-connected. Show that the map $[X, Y]_* \to [X, Y]$ from pointed to unpointed classes induces a bijection

$$[X, Y]_*/\pi_1(Y) \cong [X, Y],$$

for an appropriate action of $\pi_1(Y)$ on $[X, Y]_*$.

Hint. Consider the map $\mathrm{ev}_{x_0}\colon \mathrm{Map}(X, Y) \to Y$, whose fiber over y_0 is $\mathrm{Map}_*(X, Y)$, and use Exercise 2.1.

(b) Let G be a topological group and X a pointed connected CW complex. Show that the bijection

$$[X, BG]_* \xrightarrow{\sim} \pi_0\mathrm{Bun}_{G,*}(X), \quad [f] \mapsto [f^*(EG)],$$

is G-equivariant, where G acts on $[X, BG]_*$ via the action of $\pi_1(BG) \cong \pi_0(G)$ from (a) and on $\pi_0\mathrm{Bun}_{G,*}(X)$ by changing the trivialization over the base point.

(c) Deduce from (a) and (b) that BG represents the functor $\pi_0\mathrm{Bun}_G\colon \mathsf{hCW}^{\mathrm{op}} \to \mathsf{Set}$.

Exercise 4.3. Let hGpd denote the homotopy category of groupoids (whose morphisms are isomorphism classes of functors) and let $\mathsf{CW}_{\leq n}$ be the category of n-truncated CW complexes. Show that the fundamental groupoid construction induces an equivalence of categories

$$\mathsf{hCW}_{\leq 1} \xrightarrow{\sim} \mathsf{hGpd}, \quad X \mapsto \Pi_1(X).$$

Hint. One can use Exercises 2.6(b) and 4.2(a) to compute $[K(G,1), K(H,1)]$.

Remark. This is called the 1-*truncated homotopy hypothesis*. The general homotopy hypothesis asserts that for any $-2 \leq n \leq \infty$ there is a higher category Gpd_n of *n-groupoids* and a functor $\Pi_n\colon \mathsf{Top} \to \mathsf{Gpd}_n$ inducing an equivalence $\mathsf{hCW}_{\leq n} \simeq \mathsf{hGpd}_n$.

Exercise 4.4. Let $\mathrm{Cov}(X)$ denote the groupoid of covering spaces over X.

(a) Show that

$$\mathsf{Top}^{\mathrm{op}} \to \mathsf{Gpd}, \quad X \mapsto \mathrm{Cov}(X),$$

is homotopy invariant, in the sense that it sends homotopy equivalences to equivalences of groupoids.

Hint. Recall that, due to I being simply connected and locally connected, every covering E of $X \times I$ is isomorphic (via path lifting) to $E_0 \times I$.

(b) Fix a set S and let $\mathrm{Cov}_S(X)$ be the groupoid of covering spaces whose fibers have cardinality $|S|$. Use the Brown representability theorem to show that there exists a CW complex B_S and a covering space $E_S \in \mathrm{Cov}_S(B_S)$ such that for any CW complex X, there is a bijection

$$[X, B_S] \xrightarrow{\sim} \pi_0 \mathrm{Cov}_S(X), \quad [f] \mapsto [f^*(E_S)].$$

Hint. To apply Brown representability, define an appropriate groupoid $\mathrm{Cov}_{S,*}(X)$ for $X \in \mathsf{Top}_*$. Recall also that any homotopy pushout square is isomorphic in hTop to a pushout square of open embeddings.