SoSe 24 Algebraic Topology II
Exercise sheet 5 (due June 28)

Exercise 5.1. Let $n \geq 0$ and let E be a finite-dimensional (real or complex) vector space. Denote by $\mathrm{St}_n(E) \subset E^n$ the space of linearly independent n-tuples of vectors in E, and by $\mathrm{Gr}_n(E) = \mathrm{St}_n(E)/\mathrm{GL}_n$ the Grassmannian of n-planes in E. Show that the quotient map $\mathrm{St}_n(E) \to \mathrm{Gr}_n(E)$ is a principal GL_n-bundle.

Exercise 5.2. Let $\mathbb{N}_\leq$ denote the poset of natural numbers.

(a) Show that the colimit functor $\mathrm{colim}\colon \mathrm{Fun}(\mathbb{N}_\leq, \mathsf{Ab}) \to \mathsf{Ab}$ is exact.

(b) Show that the limit functor $\lim\colon \mathrm{Fun}(\mathbb{N}_\leq^{\mathrm{op}}, \mathsf{Ab}) \to \mathsf{Ab}$ is not exact.

(c) Construct a functor
$$\lim\nolimits^1\colon \mathrm{Fun}(\mathbb{N}_\leq^{\mathrm{op}}, \mathsf{Ab}) \to \mathsf{Ab}$$
such that, for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{Fun}(\mathbb{N}_\leq^{\mathrm{op}}, \mathsf{Ab})$, there is an induced exact sequence
$$0 \to \lim A \to \lim B \to \lim C \xrightarrow{\partial} \lim\nolimits^1 A \to \lim\nolimits^1 B \to \lim\nolimits^1 C \to 0.$$
Hint. The limit of A can be written as the kernel of a certain map $\prod_n A_n \to \prod_n A_n$. Define $\lim^1 A$ to be the cokernel of the same map.

(d) Let $A \in \mathrm{Fun}(\mathbb{N}_\leq^{\mathrm{op}}, \mathsf{Ab})$ be a tower of abelian groups such that each map $A_{n+1} \to A_n$ is surjective. Show that $\lim^1 A = 0$.

(e) Let $X \in \mathrm{Fun}(\mathbb{N}_\leq, \mathsf{CW}_*)$ be a sequence of pointed CW complexes. Show that there is for all $i \geq 0$ a short exact sequence
$$0 \to \lim_n\nolimits^1 \tilde{H}^{i-1}(X_n) \to \tilde{H}^i(\mathrm{Tel}_*(X)) \to \lim_n \tilde{H}^i(X_n) \to 0.$$
This is known as the *Milnor exact sequence*.

Hint. Write the telescope as a suitable union of two subcomplexes and consider the associated long exact sequence in cohomology.

Exercise 5.3. Let $p\colon X \to B$ be a fibration. Construct a functor
$$\Pi_1(B) \to \mathsf{hTop}, \quad b \mapsto X_b = \mathrm{fib}_b(p).$$
Deduce that for any map $p\colon X \to B$ there is a functor
$$\Pi_1(B) \to \mathsf{hTop}, \quad b \mapsto \mathrm{hfib}_b(p).$$

Exercise 5.4.

(a) Compute $H_*(\mathbb{RP}^2, \tilde{\mathbb{Z}})$ and $H^*(\mathbb{RP}^2, \tilde{\mathbb{Z}})$, where $\tilde{\mathbb{Z}}$ is the sign representation of $\pi_1(\mathbb{RP}^2)$.

(b) Let K be the Klein bottle. Compute $H_*(K, \rho)$, where $\rho\colon \pi_1(K) = \langle a, b \mid abab^{-1} \rangle \to \mathrm{Aut}(\mathbb{Z})$ is the representation given by $\rho(a) = -\mathrm{id}$ and $\rho(b) = \mathrm{id}$.

Exercise 5.5. Let A be an object of an abelian category with a filtration

$$\cdots \subset F_{t-1}A \subset F_tA \subset F_{t+1}A \subset \cdots \subset A \quad (t \in \mathbb{Z}).$$

Let $F^tA = A/F_tA$ be the associated cofiltration, and define

$$F_\infty A = \operatorname*{colim}_{t\to\infty} F_tA, \qquad F_{-\infty}A = \lim_{t\to-\infty} F_tA,$$
$$F^\infty A = \operatorname*{colim}_{t\to\infty} F^tA, \qquad F^{-\infty}A = \lim_{t\to-\infty} F^tA.$$

The filtration F_*A is called:

- *exhaustive* if $F_\infty A \to A$ is an isomorphism;
- *coseparated* if $F^\infty A = 0$;
- *complete* if $A \to F^{-\infty}A$ is an isomorphism;
- *separated* if $F_{-\infty}A = 0$.

Prove the following statements:

(a) In Ab, a filtration is exhaustive if and only if it is coseparated; every complete filtration is separated, but there exist separated filtrations that are not complete.

(b) Let F_*A and F_*B be filtrations of A and B and let $f\colon A \to B$ be a filtered map (i.e., $f(F_tA) \subset F_tB$ for all $t \in \mathbb{Z}$) such that the induced map $\mathrm{gr}_*(f)\colon \mathrm{gr}_*A \to \mathrm{gr}_*B$ is an isomorphism. If both filtrations are exhaustive and complete, then f is an isomorphism.

Exercise 5.6. Let F_*C be a filtered chain complex of abelian groups

$$\cdots \subset F_{t-1}C \subset F_tC \subset F_{t+1}C \subset \cdots \subset C.$$

Consider the associated spectral sequence with

$$\begin{aligned} E^r_{s,t} &= Z^r_{s,t}/B^r_{s,t}, \\ Z^r_{s,t} &= \text{kernel of } H_s(F_t/F_{t-1}) \xrightarrow{\partial} H_{s-1}(F_{t-1}/F_{t-r}), \\ B^r_{s,t} &= \text{image of } H_{s+1}(F_{t+r-1}/F_t) \xrightarrow{\partial} H_s(F_t/F_{t-1}). \end{aligned}$$

Let $F_tH_s(C) \subset H_s(C)$ denote the image of $H_s(F_tC) \to H_s(C)$.

(a) Show that $\mathrm{gr}_tH_s(C)$ is a subquotient of $E^\infty_{s,t}$:

$$\mathrm{gr}_tH_s(C) \twoheadleftarrow \mathrm{gr}_tH_s(F_\infty C) \hookrightarrow E^\infty_{s,t}.$$

The filtration F_*C is called *weakly convergent* when $\mathrm{gr}_tH_s(C) = E^\infty_{s,t}$. It is called *strongly convergent* if moreover the filtration $F_*H_*(C)$ is exhaustive and complete.

(b) Suppose that $F_\infty C \to C$ is a quasi-isomorphism and that, for every s, there exists t_0 such that $H_s(F_tC) = 0$ for $t \le t_0$. Show that F_*C is strongly convergent.

The point of strong convergence is the following:

(c) Let $f\colon C \to D$ be a filtered chain map inducing quasi-isomorphisms $\mathrm{gr}_tC \to \mathrm{gr}_tD$ for all t. If F_*C and F_*D are strongly convergent, then f is a quasi-isomorphism.

Hint. Use Exercise 5.5(b).