SoSe 24 Algebraic Topology II
Exercise sheet 6 (due July 12)

Exercise 6.1. Let $n \geq 2$ and let $d \neq 0$. Let F_d be the homotopy fiber of a map $S^n \to S^n$ of degree d.

(a) Using the Serre spectral sequence, compute the homology of F_d.

(b) Determine all pages and differentials, starting at E^2 and up to signs, of the homological Serre spectral sequence for the homotopy fiber sequence

$$\Omega S^n \to F_d \to S^n.$$

Hint. Recall that

$$H_i(\Omega S^n) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is a multiple of } n-1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Assume $n \geq 3$. Determine all pages and differentials, starting at E^2, of the homological Serre spectral sequence for the homotopy fiber sequence

$$\Omega S^n \to \Omega S^n \to F_d.$$

Determine the group extensions needed to assemble $H_*(\Omega S^n)$ from the E^∞ page.

Exercise 6.2. Compute all pages and differentials (starting at E^2) of the homological Serre spectral sequence for the homotopy fiber sequence

$$K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}/8, 1) \to K(\mathbb{Z}/4, 1).$$

Hint. Recall that

$$H_i(K(\mathbb{Z}/m, 1)) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ \mathbb{Z}/m, & \text{if } i > 0 \text{ is odd}, \\ 0, & \text{otherwise.} \end{cases}$$

To figure out whether $E^2_{1,3} = \mathbb{Z}/2$ is killed by d_3 or by d_4, use functoriality to compare the given Serre spectral sequence with the one for $K(\mathbb{Z}/2,1) \to K(\mathbb{Z}/4,1) \to K(\mathbb{Z}/2,1)$. You may take for granted that the map $K(\mathbb{Z}/2,1) \to K(\mathbb{Z}/4,1)$ induces the nonzero map $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ on all odd homology groups. From this comparison, deduce that all d_3's must be zero, and ultimately that $E^6 = E^\infty$.

Exercise 6.3. Let B be a CW complex and $f \colon X \to B$ a fibration. In this exercise, we identify the so-called "edge homomorphisms" of the Serre spectral sequence.

(a) Suppose that the fibers of f are connected. Show that the composition

$$H_p(X) \twoheadrightarrow H_p(X)/F_{p-1}H_p(X) \cong E^\infty_{p,0} \hookrightarrow E^2_{p,0} \cong H_p(B)$$

is $f_* \colon H_p(X) \to H_p(B)$.

(b) Suppose that B is simply connected and let $i\colon F \hookrightarrow X$ be a chosen fiber of p. Show that the composition

$$H_q(F) \cong E^2_{0,q} \twoheadrightarrow E^\infty_{0,q} \cong F_0 H_q(X) \hookrightarrow H_q(X)$$

is $i_*\colon H_q(F) \to H_q(X)$.

Exercise 6.4. Let F_*C be a strongly convergent filtration of a chain complex C (see Exercise 5.6), let $E^r_{s,t}$ be the associated spectral sequence, and let $n \geq 1$. Suppose that the E^n page is everywhere zero, except on two parallel lines of finite integral slope $e < n$ in the (s,t)-plane, separated vertically by $n - e$ units. This happens for example if the filtration F_*C only jumps twice, in which case $e = 0$ and n is the distance between the jumps.

(a) Draw a picture of the situation and note that the differential d_n maps the upper line to the lower line. Construct a long exact sequence of the form

$$\cdots \to H_{s+1}(C) \to E^n_{s+1,f(s)+n} \xrightarrow{d_n} E^n_{s,f(s)} \to H_s(C) \to \cdots,$$

where $t = f(s)$ is the equation of the lower line.

We now look at two instances of this phenomenon in the Serre spectral sequence. Let $f\colon X \to B$ be a fibration with chosen fiber $i\colon F \hookrightarrow X$.

(b) Suppose that $B \simeq S^n$ for some $n \geq 2$. Construct a long exact sequence

$$\cdots \to H_{s+1}(X) \to H_{s+1-n}(F) \to H_s(F) \xrightarrow{i_*} H_s(X) \to \cdots.$$

This is called the *Wang sequence.*

(c) Suppose that $F \simeq S^n$ for some $n \geq 1$ and that B is simply connected. Construct a long exact sequence

$$\cdots \to H_{s+1}(X) \xrightarrow{f_*} H_{s+1}(B) \to H_{s-n}(B) \to H_s(X) \to \cdots.$$

This is called the *Gysin sequence.*

(d) What changes in (b) when $n = 1$ and in (c) when B is only connected?

Exercise 6.5. Let n be odd. Show that the cohomology ring $H^*(\Omega S^{n+1})$ has the form $\Lambda_{\mathbb{Z}}(\gamma_1) \otimes \Gamma_{\mathbb{Z}}(\gamma_2)$ for some cohomology classes γ_1 in degree n and γ_2 in degree $2n$. Explicitly,

$$H^*(\Omega S^{n+1}) \cong \mathbb{Z}[\gamma_1, \gamma_{2k} \mid k \geq 1]/(\gamma_1^2, \gamma_2^k - k!\gamma_{2k}).$$

Exercise 6.6. Use the Serre spectral sequence to determine the cohomology groups $H^i(K(\mathbb{Z},3))$ for $i \leq 13$, as well as the ring structure on $H^*(K(\mathbb{Z},3))/H^{>13}(K(\mathbb{Z},3))$.

Hint. Recall that $\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) \simeq \mathbb{CP}^\infty$.