SOSE 2024 SEMINAR: DE RHAM COHOMOLOGY

The goal of this seminar is to introduce and study de Rham cohomology, which is an invariant of smooth manifolds defined using differential forms. It is related both to analysis (via the fundamental theorem of calculus and Stokes' theorem) and algebraic topology (as de Rham cohomology turns out to be isomorphic to singular cohomology). Moreover, the ideas underlying the definition of de Rham cohomology are quite versatile and can be applied in many other geometric contexts, for example in algebraic geometry.

The seminar will closely follow Chapter I of [BT82] and cover most of the results therein. The main theorem is *Poincaré duality*, which is a surprising symmetry in the cohomology of smooth compact manifolds. We will discuss it for both oriented and nonorientable manifolds.

No prior knowledge of algebraic topology is required. Basic knowledge of linear algebra and integration in several variables (Analysis III) will be assumed. Familiarity with manifolds (Analysis IV) is helpful but not strictly required, as the necessary concepts will be reviewed.

Talks in English are preferred, but talks in German are possible if all the participants agree. Reports can be written in either English or German.

(1) **The de Rham complex of** $\mathbb{R}^n$ [BT82, §1] Define the de Rham complex $\Omega^*(\mathbb{R}^n)$ and the de Rham cohomology groups $H^*(\mathbb{R}^n)$ of $\mathbb{R}^n$. Explain how the fundamental theorem of calculus translates to $H^1(\mathbb{R}^1) = 0$ [BT82, Example 1.5]. Show that a short exact sequence of complexes induces a long exact sequence on cohomology groups [BT82, p. 17, Exercise]. Define the de Rham complex and de Rham cohomology of $\mathbb{R}^n$ with compact supports. Compute $H_c^*(\mathbb{R}^1)$ [BT82, Example 1.6].
(2) **The de Rham complex of a smooth manifold** [BT82, §2] Define the de Rham complex $\Omega^*(M)$ and the de Rham cohomology groups $H^*(M)$ of a C^∞ smooth manifold M [BT82, §2, up to p. 21], as well as their versions with compact supports [BT82, §2, pp. 25–26]. Show that smooth manifolds admit partitions of unity [War83, Theorem 1.11].
(3) **The Mayer–Vietoris sequence** [BT82, §2] Derive the Mayer–Vietoris long exact sequence for de Rham cohomology and de Rham cohomology with compact supports [BT82, §2]. Use it to compute the de Rham cohomology of the circle [BT82, Example 2.6].
(4) **Orientation and integration** [BT82, §3] Define the notion of orientation of a smooth manifold and the integral of a top-dimensional form on an oriented manifold. Construct the induced orientation on the boundary of an oriented manifold. [BT82, §3, up to p. 31].
(5) **Stokes' theorem** [BT82, §3] Prove Stokes' theorem [BT82, Theorem 3.5].
(6) **The Poincaré lemma** [BT82, §4] Prove [BT82, Proposition 4.1] for a smooth manifold M. Deduce the Poincaré lemma [BT82, Corollary 4.1.1] and the homotopy axiom [BT82, Corollary 4.1.2]. Compute the de Rham cohomology of the n-sphere S^n [BT82, Exercise 4.3].
(7) **The Poincaré lemma with compact supports** [BT82, §4] Prove [BT82, Proposition 4.7] and deduce the Poincaré lemma for de Rham cohomology with compact supports [BT82, Corollary 4.7.1]. Compute the de Rham cohomology and the de Rham cohomology with compact supports of the open Möbius strip [BT82, Exercise 4.8].
(8) **Finiteness of de Rham cohomology** [BT82, §5] Introduce the notion of good cover. Show that the de Rham cohomology groups of a smooth manifold with a finite good cover are finite-dimensional $\mathbb{R}$-vector spaces [BT82, Proposition 5.3.1], and similarly with compact supports. Sketch the proof that every (compact) smooth manifold admits a (finite) good cover [BT82, Theorem 5.1].
(9) **Poincaré duality for oriented manifolds** [BT82, §5] Introduce the integration pairing on an oriented manifold M, and prove that it is nondegenerate when M admits a finite good cover [BT82, §5, pp. 44–46] (also mention [BT82, Remark 5.7]). Deduce that $H_c^n(M) \simeq \mathbb{R}$ if M is a connected oriented n-manifold [BT82, Corollary 5.8].
(10) **The degree of a proper map** [BT82, §4] Explain how a proper map $f\colon M \to N$ between smooth manifolds induces a pullback $f^*\colon H_c^*(N) \to H_c^*(M)$ on de Rham cohomology with compact supports. If M and N are oriented of the same dimension, define the degree of f using Poincaré duality [BT82, after Corollary 5.8]. State Sard's theorem for a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$

and prove its generalization to smooth manifolds [BT82, Theorem 4.11]. Deduce that the degree is always an integer [BT82, §4, p. 41].

(11) **The Künneth formula and the Leray–Hirsch theorem** [BT82, §5] Define fiber bundles and prove the Leray–Hirsch theorem [BT82, Theorem 5.11]. State the Künneth formula as a special case.

(12) **Vector bundles and the Thom isomorphism** [BT82, §6] Define vector bundles and oriented vector bundles on a smooth manifold, and give the example of the tangent bundle [BT82, Example 6.3]. State the triviality of vector bundles on $\mathbb{R}^n$ [BT82, Corollary 6.9]. For $\pi\colon E \to M$ an oriented vector bundle of rank n, define the de Rham complex $\Omega^*_{cv}(E)$ of forms with compact supports in the vertical direction, and define integration along the fibers $\pi_*\colon \Omega^*_{cv}(E) \to \Omega^{*-n}(M)$. Sketch the proof of the Thom isomorphism for an oriented vector bundle [BT82, Theorem 6.17].

(13) **The orientation bundle and the twisted de Rham complex** [BT82, §7] Define differential forms valued in a vector bundle. Define the notion of a flat vector bundle (E, ϕ) over a manifold M and the corresponding de Rham complex $\Omega^*_\phi(M, E)$ and cohomology groups $H^*_\phi(M, E)$ [BT82, §7, p. 80]. Define the orientation line bundle L and the twisted de Rham complex $\Omega^*(M, L)$.

(14) **Densities and Poincaré duality for nonorientable manifolds** [BT82, §7] Explain how to generalize integration to nonorientable manifolds using densities. State Stokes' theorem for densities [BT82, Theorem 7.7], the general Poincaré duality theorem [BT82, Theorem 7.8], and the general Thom isomorphism theorem [BT82, Theorem 7.10]. Along the way, prove [BT82, Corollary 7.8.1].

References

[BT82] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, 1982, `http://www.mathematik.ur.de/hoyois/WS21/derham/bott-tu.pdf`

[War83] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, 1983, `http://www.mathematik.ur.de/hoyois/WS21/derham/warner.pdf`