# SOSE 2025 SEMINAR: $\mathbb{A}^1$-INVARIANCE IN ALGEBRAIC GEOMETRY

An $\mathbb{A}^1$-homotopy is an algebraic analogue of a homotopy in topology, where the unit interval $[0,1]$ is replaced by the algebraic affine line $\mathbb{A}^1$. More precisely, if $X$ and $Y$ are algebraic varieties, an $\mathbb{A}^1$-homotopy between two morphisms $f, g\colon X \to Y$ is a morphism $h\colon X \times \mathbb{A}^1 \to Y$ such that $h(-,0) = f$ and $h(-,1) = g$. As in topology, it turns out that many interesting invariants of algebraic varieties are $\mathbb{A}^1$-*invariant*, i.e., they do not see the difference between $\mathbb{A}^1$-homotopic maps. An important example is étale cohomology, which is an algebro-geometric analogue of singular cohomology.

The goal of this seminar is to study some elementary $\mathbb{A}^1$-homotopical phenomena in algebraic geometry, using the lecture notes [Aso16] as our main reference. In particular, we will discuss algebraic vector bundles and symmetric bilinear forms. The main results we will obtain are the following:

(1) The $\mathbb{A}^1$-homotopical classification of vector bundles: if $X$ is a smooth affine variety, there is a bijection
$$\mathrm{Vect}_n(X) \cong [X, \mathrm{Gr}_n]_{\mathbb{A}^1}$$
between isomorphism classes of rank $n$ vector bundles on $X$ and $\mathbb{A}^1$-homotopy classes of maps to the Grassmannian $\mathrm{Gr}_n$.

(2) There is a bijection between $[\mathbb{P}^1, \mathbb{P}^1]_{*,\mathbb{A}^1}$ (the set of pointed $\mathbb{A}^1$-homotopy classes of endomorphisms of the projective line) and equivalence classes of non-degenerate symmetric bilinear forms. This is an elementary but nontrivial computation, which is the basis of an important connection between $\mathbb{A}^1$-homotopy theory and the theory of symmetric bilinear forms.

This seminar is intended for 3rd-year Bachelor students and Master students. Familiarity with category theory and some elementary background in commutative algebra (rings, modules, tensor products) will be assumed. Background in algebraic geometry is not necessary, as we will review the relevant notions.

The program below can be adjusted to fit the interest and background of the participants.

23.04 **Affine varieties, $\mathbb{A}^1$-homotopies, and the naive $\mathbb{A}^1$-homotopy category.** Define the category of affine schemes over a commutative ring $k$ [Aso16, §2.1]. Introduce the functor of points perspective. Define $\mathbb{A}^1$-homotopy invariants and discuss the examples of the topological realization of complex varieties and of the functor of units [Aso16, §2.2]. Define the naive $\mathbb{A}^1$-homotopy category and compute $[\operatorname{Spec} k, \mathrm{SL}_n]_{\mathbb{A}^1}$ [Aso16, Proposition 2.3.1.4].

30.04 **Projective modules I: basic properties and Zariski descent.** Review basic properties of finitely generated projective modules, in particular their characterization as locally free modules [Aso16, Proposition 3.2.3.5]. Discuss patching of modules and module homomorphisms [Aso16, §3.3.2]. Prove the Zariski descent property of the category of modules [Aso16, Theorem 3.3.3.2].

07.05 **Projective modules II: Serre's splitting theorem.** As motivation, state the splitting result for vector bundles over smooth manifolds [Aso16, Theorem 3.5.1.1]. Define sections of finitely generated projective modules [Aso16, §3.5.2]. Prove Serre's splitting theorem (Serre's original paper [Ser58] is a good reference for this).

14.05 **Picard groups and normal rings.** Define the Picard group $\mathrm{Pic}(R)$ of a commutative ring $R$ [Aso16, §4.1]. Prove the Unit-Pic exact sequence for integral domains [Aso16, Proposition 4.2.1.6]. Sketch the equivalence of Cartier and Weil divisors [Aso16, Proposition 4.5.2.4] and the $\mathbb{A}^1$-invariance of $\mathrm{Pic}(-)$ [Aso16, Theorem 4.6.1.3(1)].

21.05 **$\mathbb{A}^1$-invariance of vector bundles I: the Quillen–Suslin theorem.** If $k$ is a field (more generally, a principal ideal domain), every finitely generated projective module over the polynomial ring $k[x_1, \ldots, x_n]$ is free. This theorem was proved independently by Quillen and Suslin in 1976. Introduce the ring $R\langle t\rangle$ and its properties [Lam06, IV, §1]. Sketch one of the many proofs of Horrocks theorem [Lam06, IV, §2], and sketch Quillen's patching theorem [Lam06, V, Theorem 1.6]. Combine these results to deduce the Quillen–Suslin theorem [Lam06, V, 2.2 and 2.9] (alternatively, another self-contained treatment of Quillen–Suslin is [Lan02, XXI, §3]).

28.05 **Grothendieck topologies and sheaves.** Define the notion of Grothendieck topology on a category $\mathcal{C}$ [MM92, III, Definition 1]. Define the category of sheaves on $\mathcal{C}$ and show that it is a left exact localization of the category of presheaves [MM92, III, §4-5]. For sheaves over a topological space, show that isomorphisms of sheaves are detected on stalks.

04.06 **Schemes, functors of points, and algebraic vector bundles.** Recall the definition of schemes as locally ringed spaces [Aso16, §7.2]. Recall the facts that the category of affine schemes (as defined in the first talk) embeds fully faithfully in the category of schemes and that the latter embeds fully faithfully in the category of functors from commutative rings to sets [EH00, §VI.1]. Define projective space $\mathbb{P}^n$ as an example of a non-affine scheme and describe its functor of points [EH00, Theorem III-37]. Define the category of quasi-coherent sheaves on a scheme [Har77, II, §5], and prove that the category of quasi-coherent sheaves on Spec $R$ is equivalent to the category of $R$-modules [Har77, II, Corollary 5.5]. Sketch the classification of vector bundles on $\mathbb{P}^1$ [Aso16, §8.1.1], and explain the failure of $\mathbb{A}^1$-invariance of vector bundles in general [Aso16, Example 8.1.3.2].

11.06 **Smooth and étale morphisms.** Define the module of Kähler differentials $\Omega_{B/A}$ of a ring homomorphism $A \to B$ [Aso16, Definition 6.2.3.1, Propositions 6.3.1.3, 6.3.4.1]. Define smooth and étale ring homomorphisms [Aso16, §6.2.1, 6.2.2, 6.7.2], and state the equivalent characterization via formal smoothness [CRing, §17, Definition 2.1]. Prove the local structure theorem for smooth morphisms [CRing, §17, Theorem 3.24].

18.06 **$\mathbb{A}^1$-invariance of vector bundles II: Lindel's theorem and the Bass–Quillen conjecture.** Prove Roitman's theorem [Aso16, Theorem 8.4.1.3] and deduce the Bass–Quillen conjecture for localizations of polynomial algebras. Explain the statement of Lindel's presentation lemma [Aso16, Proposition 7.6.2.1(2)], and use it to prove the Bass–Quillen conjecture for smooth algebras over a field (Lindel's theorem) [Aso16, §8.4.2].

25.06 (No meeting.)

02.07 **The $\mathbb{A}^1$-homotopical classification of algebraic vector bundles.** Define the Grassmannian $\mathrm{Gr}_{n,k}$ as a scheme [EH00, III.2.7] and via its functor of points [EH00, VI-18]. Prove that for $X$ a smooth affine scheme over a field, there is a bijection between rank $n$ vector bundles over $X$ and $\mathbb{A}^1$-homotopy classes of morphisms $X \to \mathrm{Gr}_n$ [Aso16, §8.5.3].

09.07 **Symmetric bilinear forms and Grothendieck–Witt groups.** Define the Grothendieck–Witt group $\mathrm{GW}(R)$ of a commutative ring $R$ and give the examples of $\mathrm{GW}(\mathbb{C})$, $\mathrm{GW}(\mathbb{R})$, and $\mathrm{GW}(\mathbb{F}_q)$ [Aso16, §9.2.1]. Prove Witt's cancellation theorem [HM73, I, §4] and sketch the standard presentation of $\mathrm{GW}(k)$ for $k$ a field of characteristic $\neq 2$ ([HM73, IV, Lemma 1.1] or [Sch85, Chapter 2, Corollary 9.4]).

16.07 **$\mathbb{A}^1$-homotopy classes of endomorphisms of the projective line.** Describe pointed endomorphisms of $\mathbb{P}^1$ in terms of rational functions [Aso16, §9.1.1]. Explain the relationship between rational functions and symmetric bilinear forms via the Bézout form [Aso16, §9.1.3]. Sketch Cazanave's computation of the set of $\mathbb{A}^1$-homotopy classes of endomorphisms of $\mathbb{P}^1$ [Caz09].

23.07 **Outlook.** The Morel–Voevodsky $\mathbb{A}^1$-homotopy category.

## References


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