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Seminar $\mathbb{A}^1$-invariance in algebraic geometry

The $\mathbb{A}^1$-homotopical classification of algebraic vector bundles

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1 Introduction

1.1 Overview

This report accompanies a talk given as part of a seminar on $\mathbb{A}^1$-invariance in algebraic geometry, held in the summer term 2025 at the University of Regensburg. In the previous talk, we saw that by Quillen–Suslin–Lindel Theorem (see [Aso19, Thm. 8.4.3.1]) the functor

$$\mathrm{Vect}_n\colon \quad \mathsf{AffSm}_k^{\mathrm{op}} \to \mathsf{Set},$$

which assigns to a smooth affine scheme X over a base field k the set of isomorphism classes $\mathrm{Vect}_n(X)$ of rank n algebraic vector bundles on X and is given on morphisms by pullback, is $\mathbb{A}^1$-invariant. The goal of the present talk is to classify isomorphism classes of rank n algebraic vector bundles on a smooth affine scheme X over k in terms of morphisms from X to a suitable Grassmannian Gr_n up to naive $\mathbb{A}^1$-homotopy.

In algebraic topology isomorphism classes of real topological vector bundles of a given rank n on a paracompact topological space X are up to homotopy precisely given by pullbacks of a certain canonical real vector bundle $E_n \to G_n$ to X.[1] This space G_n is constructed as the filtered colimit over all finite-dimensional Grassmannian manifolds $G_n(\mathbb{R}^N)$ with $n \leq N$, where the transition map $N \rightsquigarrow N+1$ is induced by the inclusion $\mathbb{R}^N \hookrightarrow \mathbb{R}^{N+1}$ into the first N entries.

There turn out to be many parallels between the topological and the algebro-geometric settings. Accordingly, we begin in section 2 by studying Grassmannians as objects in scheme theory. After giving a concrete definition of finite-dimensional Grassmannian schemes in 2.1, we proceed in 2.2 and 2.3 towards understanding the functor of points represented by a Grassmannian scheme. For an affine scheme $\mathrm{Spec}(R)$ this Grassmannian functor parametrizes direct summands of R^n of a fixed rank, which generalizes the classical definition of Grassmannians when R is a field. In the spirit of the seminar, we focus hereby on the affine case, though many results hold more generally.
In close analogy with the topological construction, the ∞-Grassmannians Gr_n will be then defined in 2.4 as the filtered colimit over all finite-dimensional Grassmannians with fixed parameter n. Here the functor of points perspective is essential, since this colimit exists only in the category $\mathsf{PSh}(\mathsf{Aff})$ of Set-valued presheaves on the category of affine schemes Aff.
Next we revisit in 3.1 the notion of naive $\mathbb{A}^1$-homotopies introduced in earlier talks and extend it to the setting relevant for this paper, where the target of the morphism is an arbitrary presheaf on affine schemes. In 3.2 we finally prove the above mentioned $\mathbb{A}^1$-homotopical classification theorem for algebraic vector bundles.

1.2 Literature

The primary source for the introduction to Grassmannian schemes in Section 2 is [EH00]. For the main result of this report - the classification of algebraic vector bundles up to naive $\mathbb{A}^1$-homotopy in Section 3 - we follow [Aso19].
Basic facts from algebraic geometry are referenced throughout from [GW10] and [Stacks]. Some results on exterior powers from commutative algebra are recalled from [Bou74].

[1]See for example [Hat17, Thm. 1.16].

2 Grassmannians

2.1 Grassmannians by gluing affine charts

Notation 2.1.1. Let R be a ring and $n, m \in \mathbb{N}$ with $1 \leq m \leq n$. Given a matrix $M \in \mathrm{Mat}_{m\times n}(R)$ and an index set $I = \{i_1 < \cdots < i_m\} \subseteq \{1, \ldots, n\}$, we denote by $M_I := (M_{i,i_k})_{1\leq i,k\leq m}$ the I-th maximal quadratic submatrix of M.

Motivation 2.1.2 (classical Grassmannian varieties)**.** Let K be a field and $n, N \in \mathbb{N}$ natural numbers with $1 \leq n < N$. In its simplest form, the Grassmannian $G_n(K^N)$ is defined as the set of n-dimensional linear subspaces of K^N. This set admits a structure of a projective variety. For a comprehensive treatment of this standard result from classical algebraic geometry, the reader is referred to [Gat24, Ch. 8]. In order to motivate the subsequent definition of the Grassmannians in the scheme theoretic context, which in fact mirrors the classical construction, we outline the main ideas.
The most natural way to view $G_n(K^N)$ as a variety is by identifying it with its image under the so called *Plücker embedding*

$$G_n(K^N) \to \mathbb{P}(\textstyle\bigwedge^n(K^N)), \qquad \mathrm{Span}_K(v_1, \ldots, v_n) \mapsto [v_1 \wedge \cdots \wedge v_n],$$

where $\bigwedge^n(K^N)$ denotes the *n-th exterior power of* K^N. It follows from linear algebra that the Plücker embedding is well-defined and injective and its image is cut out by polynomial equations. The vector space $\bigwedge^n(K^N)$ admits a canonical basis consisting of the tensors $e_I := e_{i_1} \wedge \cdots \wedge e_{i_n}$, indexed by subsets $I = \{i_1 < \cdots < i_n\} \subseteq \{1, \ldots, N\}$, where e_i is the i-th standard basis vector of K^N. The e_I-coordinate of a vector $v_1\wedge\cdots\wedge v_n$ is exactly the I-th maximal minor of the $n\times N$-matrix with rows $v_1, \ldots, v_n$. The standard cover of $\mathbb{P}(\bigwedge^n(K^N))$ by the affine open subsets

$$U_I := \{[x] \in \mathbb{P}(\textstyle\bigwedge^n(K^N)) \mid e_I - \text{coordinate of } x \text{ is nonzero}\}$$

gives rise to an affine open cover $(G_n(K^N) \cap U_I)_I$ of $G_n(K^N)$. For later on, it is useful to obtain a more explicit description of these open subsets. Denote by $\mathrm{Mat}^{\max}_{n\times N}(K)$ the set of $n \times N$-matrices over K of maximal rank. Mapping such a matrix to the linear subspace of K^N spanned by its rows, yields a well-defined bijection

$$f\colon\ \mathrm{Mat}^{\max}_{n\times N}(K)/\,\mathrm{GL}_n(K) \xrightarrow{\sim} G_n(K^N),$$

where the quotient on the left-hand side is taken with respect to the natural left-action of $\mathrm{GL}_n(K)$ on $\mathrm{Mat}^{\max}_{n\times N}(K)$ by left-multiplication. Given $I \subseteq \{1, \ldots, N\}$ with $\#I = n$, the map f restricts to a bijection

$$\{[M] \in \mathrm{Mat}^{\max}_{n\times N}(K)/\,\mathrm{GL}_n(K) | \det(M_I) \neq 0\} \xrightarrow{\sim} G_n(K^N) \cap U_I.$$

On the left-hand side left-multiplication with M_I^{-1} yields a unique representative of $[M]$, which has the identity matrix as its I-th submatrix. We thus obtain a bijection

$$f_I\colon K^{n(N-n)} \cong \{M \in \mathrm{Mat}^{\max}_{n\times N}(K) \mid M_I = E_n\} \xrightarrow{\sim} G_n(K^N) \cap U_I.$$

In fact this is even an isomorphism of varieties: The coordinates of the image of an $n \times (N - n)$-matrix M under f_I are given as the maximal minors of the $n \times N$-matrix obtained by expanding M by the $n \times n$-identity matrix E_n at the I-th columns. The inverse is given by multiplying the matrix, whose rows consist of a basis of the given n-dimensional subspace of K^N, with the inverse of its I-th submatrix and afterwards deleting the columns indexed by I. Both operations are defined by polynomial functions. Thus, in the classical setting, the Grassmannian variety $G_n(K^N)$ admits an open cover by charts isomorphic to affine $n(N-n)$-spaces. In the language of schemes, this description can be directly replicated via gluing and will serve as our definition of the Grassmannians as a scheme.

The upcoming Lemma formally establishes the technique of *gluing schemes*. In this, we follow [GW10, Section (3.5)].

Definition 2.1.3 (gluing datum of schemes)**.** For an index set I we define a *gluing datum of schemes* as a tripple $((U_i)_{i\in I}, (U_{i,j})_{i,j\in I}, (\varphi_{i,j})_{i,j\in I})$, consisting of a family of schemes $(U_i)_{i\in I}$ and for all $i, j \in I$ an open subscheme $U_{i,j} \subseteq U_i$ and an isomorphism $\varphi_{j,i} \colon U_{i,j} \to U_{j,i}$, such that the following conditions are satisfied:

(1) For all $i \in I$ we have $U_{i,i} = U_i$.

(2) For all $i, j, k \in I$ we have $\varphi_{j,i}(U_{i,j} \cap U_{i,k}) \subseteq U_{j,k}$.

(3) For all $i, j, k \in I$ the *cocycle condition* $(\varphi_{k,j} \circ \varphi_{j,i})|_{U_{i,j}\cap U_{i,k}} = \varphi_{k,i}|_{U_{i,j}\cap U_{i,k}}$ holds.

Lemma 2.1.4 (gluing schemes)**.** Let $((U_i)_{i\in I}, (U_{i,j})_{i,j\in I}, (\varphi_{i,j})_{i,j\in I})$ be a gluing datum of schemes. There exists a pair $(X, (\psi_i)_{i\in I})$, consisting of a scheme X and for all $i \in I$ an open immersion $\psi_i \colon U_i \to X$, such that:

- The family $(\psi_i(U_i))_{i\in I}$ covers X.
- For all $i, j \in I$ we have $\psi_j \circ \varphi_{j,i} = \psi_i|_{U_{i,j}}$.
- For all $i, j \in I$ we have $\psi_i(U_i) \cap \psi_j(U_j) = \psi_i(U_{i,j})$.

The pair $(X, (\psi_i)_{i\in I})$ is unique up to unique isomorphism.

Proof. See [GW10, Prop. 3.10]. □

One of the two equivalent definitions of Grassmannian schemes in [EH00, III.2.7] mainly capitalizes on the following technical lemma.

Lemma 2.1.5. Let $n, N \in \mathbb{N}$ with $1 \le n < N$. Let A be a ring and consider the ring $S := A[x_{i,j} \mid 1 \le i \le n, 1 \le j \le N]$ such that $\operatorname{Spec} S = \mathbb{A}_A^{nN}$. Write

$$X := (x_{i,j})_{1\le i\le n, 1\le j\le N} \in \mathrm{Mat}_{n\times N}(S).$$

For subsets $I = \{i_1 < \cdots < i_n\}, J = \{j_1 < \cdots < j_n\} \subseteq \{1, \ldots, N\}$ consider the closed subschemes

$$W_I := V(x_{i,i_k} - \delta_{i,k} \mid 1 \le i, k \le n) \subseteq \mathbb{A}_A^{nN}$$

and the open subschemes

$$V_J := D(\det(X_J)) \subseteq \mathbb{A}_A^{nN}.$$

Define $W_{I,J} := W_I \cap V_J$. Then the morphism of rings

$$\mu_{J,I}\colon (S/(x_{i,j_k} - \delta_{i,k} \mid 1 \le i,k \le n))_{\det(X_I)} \xrightarrow{\sim} (S/(x_{i,i_k} - \delta_{i,k} \mid 1 \le i,k \le n))_{\det(X_J)}$$

given by "$X \mapsto X_J^{-1}X$" (by abuse of notation of X for $([x_{i,j}])_{1\le i\le n, 1\le j\le N}$) is an isomorphism. The induced isomorphism

$$\varphi_{J,I} := \operatorname{Spec}(\mu_{J,I})\colon W_{I,J} \xrightarrow{\sim} W_{J,I}$$

gives rise to a gluing datum of schemes $((W_I)_I, (W_{I,J})_{I,J}, (\varphi_{I,J})_{I,J})$. Moreover, for any $I \subseteq \{1,\dots,N\}$ with $\#I = n$, there exists a canonical isomorphism $\mathbb{A}_A^{n(N-n)} \xrightarrow{\sim} W_I$.

Definition 2.1.6 (Grassmannian schemes)**.** Let $n, N \in \mathbb{N}$ with $1 \le n < N$.

(i) Let A be a ring. Then the *Grassmannian scheme* $\operatorname{Gr}_A(n,N)$ is defined as the scheme, which is obtained by Lemma 2.1.4 from the respective gluing datum constructed in Lemma 2.1.5.

(ii) Let S be a scheme. Then we define the *Grassmannian scheme* $\operatorname{Gr}_S(n,N)$ as the fibered product $\operatorname{Gr}_{\mathbb{Z}}(n,N) \times_{\mathbb{Z}} S$.

After recalling basic properties of the *functor of points of a scheme* in 2.2, we will return to this very concrete, though at times technical, definition in Example 2.2.5 from a slightly different perspective and observe that some aspects of it appear more intuitively then.

Remark 2.1.7. Let $n, N \in \mathbb{N}$ with $1 \le n < N$. It is easy to verify that for any ring A, the two definitions $\operatorname{Gr}_A(n,N)$ and $\operatorname{Gr}_{\operatorname{Spec}(A)}(n,N)$ from Definition 2.1.6 agree up to unique isomorphism: Let $I = \{i_1 < \dots < i_n\}, J = \{j_1 < \dots < j_n\} \subseteq \{1,\dots,N\}$ be index sets and write W_I^A and $W_{I,J}^A$ for the affine schemes constructed in Lemma 2.1.5. The open subschemes $W_I^{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}(A)$ cover $\operatorname{Gr}_{\operatorname{Spec}(A)}(n,N)$ (see [GW10, Cor. 4.19]) and we have

$$\begin{aligned} W_I^{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}(A) &\cong \operatorname{Spec}((\mathbb{Z}/(x_{i,j_k} - \delta_{i,k} \mid 1 \le i,k \le n)) \otimes_{\mathbb{Z}} A) \\ &\cong \operatorname{Spec}(A/(x_{i,j_k} - \delta_{i,k} \mid 1 \le i,k \le n)) = W_I^A \end{aligned}$$

and analogously $W_{I,J}^{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec}(A) \cong W_{I,J}^A$ by unique isomorphisms. Since these isomorphisms are compatible with the gluing morphisms from Lemma 2.1.5, the statement follows from the uniqueness part in Lemma 2.1.4.

Proof of Lemma 2.1.5. Let $I = \{i_1 < \dots < i_n\} \subseteq \{1,\dots N\}$ be an index set. There exists a unique bijection $\sigma_I\colon \{1,\dots,N\} \setminus I \xrightarrow{\sim} \{1,\dots,N-n\}$ of ordered sets. This induces a surjective morphism of rings

$$\mu_I\colon S \to A[t_{i,j} \mid 1 \le i \le n, 1 \le j \le N-n]$$

given by

$$x_{i,j} \mapsto \begin{cases} \delta_{i,k}, & \text{if } \exists k \in \{1,\dots,n\} : j = i_k, \\ t_{i,\sigma_I(j)}, & \text{otherwise} \end{cases}$$

with kernel $\mathfrak{a}_I := (x_{i,i_k} - \delta_{i,k} \mid 1 \le i,k \le n)$.

This yields the desired isomorphism of schemes

$$\mathbb{A}_A^{n(N-n)} \xrightarrow{\sim} \operatorname{Spec}(S/\mathfrak{a}_I) = W_I.$$

Let $J \subseteq \{1, \dots, N\}$ with $\#J = n$. We now construct the morphism $\mu_{J,I}$. Write shortly d_J for $\det(X_J) \in S$. Since d_J is invertible in S_{d_J}, so is the matrix X_J over S_{d_J}. Hence "$X \mapsto X_J^{-1}X$", which we read as $x_{i,j} \mapsto (X_J^{-1}X)_{i,j}$ for all $(i,j) \in \{1, \dots, n\} \times \{1, \dots, N\}$, defines an endomorphism $f_{J,I}$ of S_{d_J}. Let $k \in \{1, \dots, n\}$. We have

$$f_{J,I}(x_{i,j_k} - \delta_{i,k}) = (X_J^{-1}X)_{i,j_k} - \delta_{i,k} = (E_n)_{i,k} - \delta_{i,k} = 0.$$

Hence by exactness of localization and since $d_J \equiv 1 \mod \mathfrak{a}_J$, we get a unique induced morphism

$$\begin{array}{ccc} S_{d_J} & \xrightarrow{f_{J,I}} & S_{d_J} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \\ S_{d_J}/(\mathfrak{a}_J)_{d_J} & & \\ \| & & \uparrow{\scriptstyle \exists !} \\ (S/\mathfrak{a}_J)_{d_J} & & \\ \| & & \\ S/\mathfrak{a}_J & \dashrightarrow & \end{array}$$

Composing with the projection

$$S_{d_J} \to S_{d_J}/(\mathfrak{a}_I)_{d_J} \cong (S/\mathfrak{a}_I)_{d_J}$$

gives a map $\widetilde{f}_{J,I} \colon S/\mathfrak{a}_J \to (S/\mathfrak{a}_I)_{d_J}$. It remains to verify that d_I is mapped to an invertible element under $\widetilde{f}_{J,I}$. We have

$$\begin{aligned} \widetilde{f}_{J,I}([d_I]) &= [\det((X_J^{-1}X)_I)] = [\det(X_J^{-1}X_I)] = [\det(X_J^{-1})\det(X_I)] \\ &= [d_J]^{-1} \cdot 1 \in (S/\mathfrak{a}_I)_{d_J}^{\times}. \end{aligned}$$

By the universal property of localization we obtain the desired map

$$\mu_{J,I} \colon (S/\mathfrak{a}_J)_{d_I} \to (S/\mathfrak{a}_I)_{d_J},$$

which corresponds to the morphism

$$\varphi_{J,I} := \operatorname{Spec}(\mu_{J,I}) \colon W_{I,J} \to W_{J,I}$$

of schemes. We have to verify that the conditions from Definition 2.1.3 are satisfied. First of all, $d_I \equiv 1 \mod \mathfrak{a}_I$ ensures (1), which we have already seen above. Moreover, it is clear by definition that $\mu_{I,I}$ is just the identity on $S/\mathfrak{a}_I$. Choose another index set $K \subseteq \{1, \dots, N\}$ with $\#K = n$. Condition (2) corresponds to the statement that the composition

$$(S/\mathfrak{a}_J)_{d_I} \xrightarrow{\mu_{J,I}} (S/\mathfrak{a}_I)_{d_J} \xrightarrow{\text{"localization"}} ((S/\mathfrak{a}_I)_{d_J})_{d_K}$$

factors through the localization map $(S/\mathfrak{a}_J)_{d_I} \to ((S/\mathfrak{a}_J)_{d_I})_{d_K}$.

This is true, since under the above map

$$[d_K] \mapsto [\det((X_J^{-1}X)_K)] = [\det(X_J^{-1}X_K)] = [\det(X_J^{-1})\det(X_K)] \\ = [d_J]^{-1}[d_K] \in ((S/\mathfrak{a}_I)_{d_J})^{\times}_{d_K}.$$

The cocycle condition (3) now corresponds to the commutativity of the diagram

$$\begin{array}{ccc} ((S/\mathfrak{a}_K)_{d_J})_{d_I} & \xrightarrow{\mu_{K,J}} & ((S/\mathfrak{a}_J)_{d_K})_{d_I} \\ & \searrow^{\mu_{K,I}} \quad \swarrow_{\mu_{J,I}} & \\ & ((S/\mathfrak{a}_I)_{d_J})_{d_K} & \end{array}$$

This is fulfilled, since in "matrix notation", the composition $(\mu_{J,I} \circ \mu_{K,J})|_{((S/\mathfrak{a}_K)_{d_J})_{d_I}}$ in the diagram is determined by

$$X \overset{\mu_{K,J}}{\longmapsto} X_K^{-1}X \overset{\mu_{J,I}}{\longmapsto} (X_J^{-1}X)_K^{-1}(X_J^{-1}X) = (X_J(X_J^{-1}X)_K)^{-1}X \\ = ((X_JX_J^{-1})X_K)^{-1}X = X_K^{-1}X,$$

which exactly describes the morphism $\mu_{K,I}$. Since we already have seen that $\varphi_{I,I}$ is the identity on W_I, the cocycle condition for $K := I$ yields by symmetry that $\varphi_{J,I}$ is an isomorphism with inverse $\varphi_{I,J}$. This finishes the proof. □

Remark 2.1.8. In [EH00, III.2.7] it is also explained how one can characterize the Grassmannian scheme $\mathrm{Gr}_A(n, N)$ for a ring A and $n, N \in \mathbb{N}$ with $1 \leq n < N$ in a more intrinsic way. Consider the ring $S := A[X_I \mid I \subseteq \{1, \ldots, N\}$ with $\#I = n]$ and define the *Plücker ideal* J as the kernel of the ringhomomorphism

$$S \to A[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq N], \qquad X_I \mapsto \det((x_{i,j})_{1\leq i\leq n, 1\leq j\leq N})_I.$$

As the name suggests, this reflects the classical construction in Motivation 2.1.2. There the coordinates of a subspace with respect to the Plücker embedding are given by the maximal minors of a maximal rank $n \times N$-matrix. If we view the variables X_I in S as the I-th maximal submatrix of such a matrix, the Plücker ideal J encodes precisely the relations between the maximal minors, which are known from determinant calculus. The ideal J is homogenous, so we may view the Grassmannians as the projective scheme

$$\mathrm{Proj}(S/J) \hookrightarrow \mathrm{Proj}\, S = \mathbb{P}_A^{\binom{N}{n}-1}.$$

In order to show that this leads to the same definition of the Grassmannian scheme as Definition 2.1.6, one covers $\mathrm{Proj}(S/J)$ by the affine opens $D_+(X_I) \cap \mathrm{Proj}(S/J)$. There exist canonical isomorphisms $W_I \xrightarrow{\sim} D_+(X_I) \cap \mathrm{Proj}(S/J)$ of affine schemes, which match the conditions in Lemma 2.1.4. We omit the details, since this global characterization of the Grassmannian scheme will not play a role later on.

2.2 Functor of points and a representability criterion

This subsection begins by revisiting the functor of points of a scheme and some basic related results. At the end we introduce a criterion for when a Set-valued presheaf on Aff is representable by a scheme. We start by stating the all-important *Yoneda lemma.*

Definition 2.2.1 (Yoneda embedding)**.** Let $\mathcal{C}$ be a locally small category. The functor

$$\begin{aligned} h\colon \quad \mathcal{C} &\to \mathsf{PSh}(\mathcal{C}), \\ C &\mapsto h_C := \mathrm{Hom}_{\mathcal{C}}(-, C), \\ (f\colon C \to D) &\mapsto (f_*\colon h_C \to h_D, \quad g \mapsto f \circ g) \end{aligned}$$

is called the *Yoneda embedding.*

Proposition 2.2.2 (Yoneda lemma)**.** Let $\mathcal{C}$ be a locally small category.

(i) Let C be an object of $\mathcal{C}$ and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ a Set-valued presheaf on $\mathcal{C}$. The map of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(h_C, F) &\to F(C), \\ \alpha &\mapsto \alpha_C(\mathrm{id}_C) \end{aligned}$$

is a bijection and natural in C and F. The inverse map assigns to an element $x \in F(C)$ the natural transformation α^x, which is valued at an object $D \in \mathcal{C}$ given by

$$\alpha^x_D\colon h_C(D) \to F(D), \qquad (f\colon D \to C) \mapsto F(f)(x).$$

(ii) The Yoneda embedding is fully faithful.

Proof. For (i) one checks that the two given maps are inverse and (ii) follows directly from (i), see for example [EH00, Lemma VI-1]. □

Notation 2.2.3 (functor of points)**.** Let F be either a presheaf on Sch or Aff. We write F^{aff} for the functor $F|_{\mathsf{Aff}}$ in $\mathsf{PSh}(\mathsf{Aff})$ and F^* for the functor $F \circ \mathrm{Spec}\colon \mathsf{CRing} \to \mathsf{Set}$. By abuse of notation, we call for a scheme X, depending on the context, the functor h_X (resp. h_X^{aff} or h_X^*) the *functor of points of* X.

The following proposition expresses the fact that schemes are built up from affine schemes in terms of the Yoneda embedding.

Proposition 2.2.4. The functor

$$h^{\mathsf{aff}}\colon \mathsf{Sch} \to \mathsf{PSh}(\mathsf{Aff}), \qquad X \mapsto h_X^{\mathsf{aff}},$$

and equivalently the functor

$$h^*\colon \mathsf{Sch} \to \mathrm{Fun}(\mathsf{CRing}, \mathsf{Set}), \qquad X \mapsto h_X^*,$$

is fully faithful.

Proof. This is a consequence of the fact that a scheme X is already determined by the open immersions from affine schemes into X. A reference is [EH00, Prop. VI-2]. □

When returning to the definition of the Grassmannian scheme via Lemma 2.1.5, it becomes clear why the functor of points perspective is a very natural way of thinking about schemes.

Example 2.2.5 (Grassmannian schemes revisited)**.** Let $n, N \in \mathbb{N}$ with $1 \leq n < N$ and $S := \operatorname{Spec} \mathbb{Z}[x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq N]$. We use the notation from Notation 2.1.1 and the proof of Lemma 2.1.5. Given index sets $I, J \subseteq \{1, \ldots, N\}$ with $\#I = \#J = n$, consider the functors

$$F_I \colon \mathsf{CRing} \to \mathsf{Set}, \qquad R \mapsto \{M \in \operatorname{Mat}_{n\times N}(R) \mid M_I = E_n\}$$

and

$$F_{I,J} \colon \mathsf{CRing} \to \mathsf{Set}, \qquad R \mapsto \{M \in \operatorname{Mat}_{n\times N}(R) \mid M_I = E_n \text{ and } \det(M_J) \in R^\times\}$$

with the obvious assignments for morphisms, where we use that a morphism of rings sends units to units. It follows from the universal property of the rings $S/\mathfrak{a}_I$ and $(S/\mathfrak{a}_I)_{d_J}$ that for a ring R the maps

$$\begin{aligned} h^*_{W_I}(R) = \operatorname{Hom}_{\mathsf{CRing}}(S/\mathfrak{a}_I, R) &\to F_I(R), \\ f &\mapsto (f([x_{i,j}]))_{1\leq i\leq n, 1\leq j\leq N} \end{aligned}$$

and

$$\begin{aligned} h^*_{W_{I,J}}(R) = \operatorname{Hom}_{\mathsf{CRing}}((S/\mathfrak{a}_I)_{d_j}, R) &\to F_{I,J}(R), \\ f &\mapsto (f([x_{i,j}]))_{1\leq i\leq n, 1\leq j\leq N} \end{aligned}$$

are bijections and natural in R. The diagram

$$\begin{array}{ccc} h^*_{W_I}(R) & \xrightarrow{\sim} & F_I(R) \\ \uparrow & & \uparrow \\ h^*_{W_{I,J}}(R) & \xrightarrow{\sim} & F_{I,J}(R) \end{array}$$

commutes. Moreover, the map

$$F_{I,J}(R) \to F_{J,I}(R), \qquad M \mapsto M_J^{-1} \cdot M$$

is bijective and natural in R. Recalling how the morphism $\mu_{J,I} \colon (S/\mathfrak{a}_J)_{d_I} \to (S/\mathfrak{a}_I)_{d_J}$ from Lemma 2.1.5 was given by the rule "$X \mapsto X_J^{-1}X$", we see that it precicely induces the composition

$$h^*_{W_{I,J}} \xrightarrow{\sim} F_{I,J} \xrightarrow{\sim} F_{J,I} \xrightarrow{\sim} h^*_{W_{J,I}}.$$

The conditions for gluing from Definition 2.1.3 now translate via the functors of points and the Yoneda lemma naturally into basic facts from matrix calculus.

A central term for the rest of the report is the *representability* of functors $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ or $\mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$. Often, we are also interested in "representing" a functor in $\mathsf{PSh}(\mathsf{Aff})$ by a (not necessarily affine) scheme via restriction of the functor of points.

Definition 2.2.6 (in Sch representable functors)**.** Let $F \in \mathsf{PSh}(\mathsf{Aff})$.

(i) A *representation of* F *in* Sch is a pair (X, α) consisting of a scheme X and an isomorphism of functors $h_X^{\mathsf{aff}} \xrightarrow{\sim} F$. We say that X *represents* F *in* Sch.

(ii) The presheaf F is called *representable in* Sch if there exists a representation of F in Sch.

In the same way as representations of a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ by an object in $\mathcal{C}$ are unique, this also holds for in Sch representable presheaves on affine schemes as a consequence of Proposition 2.2.4.

Proposition 2.2.7 (uniqueness of representations in Sch)**.** Given an in Sch representable presheaf $F \in \mathsf{PSh}(\mathsf{Aff})$ with representations (X, α) and (Y, β), there exists a unique isomorphism $f\colon X \xrightarrow{\sim} Y$ such that $\beta \circ h^{\mathsf{aff}}(f) = \alpha$.

Proof. The natural isomorphism $\gamma := \beta^{-1} \circ \alpha\colon h_X^{\mathsf{aff}} \to h_Y^{\mathsf{aff}}$ is unique with $\beta \circ \gamma = \alpha$. By Proposition 2.2.4 the restricted Yoneda embedding h^{aff} is fully faithful. Thus there exist unique $f\colon X \to Y$ and $g\colon Y \to X$ such that $h^{\mathsf{aff}}(f) = \gamma$ and $h^{\mathsf{aff}}(g) = \gamma^{-1}$. Since

$$h^{\mathsf{aff}}(g \circ f) = \mathrm{id}_{h_X^{\mathsf{aff}}} \quad \text{and} \quad h^{\mathsf{aff}}(f \circ g) = \mathrm{id}_{h_Y^{\mathsf{aff}}}$$

by functoriality, we get $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ using that h^{aff} is faithful. Hence f is an isomorphism and unique with the property $\beta \circ h^{\mathsf{aff}}(f) = \alpha$. □

We conclude that a scheme X is uniquely determined by its functor of points on Aff. It still remains the question, when a functor $\mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ can be represented in Sch. With the criterion in [EH00, Thm. VI-14], we attain, after going through a few basic definitions, a suitable criterion, which allows us in the upcoming section to determine the functor represented by Grassmannians.

Notation 2.2.8. Let F be a presheaf on (affine) schemes and $\iota\colon U \hookrightarrow X$ the inclusion of an (affine) open subscheme U into an (affine) scheme X. Let $f \in F(X)$. Then we write $f|_U$ for the element $F(\iota)(f)$ in $F(U)$.

Definition 2.2.9 (Zariski sheaf)**.** A *Zariski sheaf on* Aff is a sheaf on the big affine Zariski site, i.e. a presheaf $F \in \mathsf{PSh}(\mathsf{Aff})$ such that for every affine scheme X and affine open covering $(U_i)_{i \in I}$ of X the diagram

$$F(X) \xrightarrow[f \mapsto (f|_{U_i})_{i \in I}]{} \prod_{i \in I} F(U_i) \overset{(f_i)_{i \in I} \mapsto (f_i|_{U_i \cap U_j})_{(i,j) \in I^2}}{\underset{(f_i)_{i \in I} \mapsto (f_j|_{U_i \cap U_j})_{(i,j) \in I^2}}{\rightrightarrows}} \prod_{(i,j) \in I^2} F(U_i \cap U_j)$$

is an equalizer diagram.

The following lemma is essential when dealing with presheaves valued in Set.

Lemma 2.2.10 ((co)limits of presheaves)**.** Let $\mathcal{C}$ be a category. The category $\mathrm{Fun}(\mathcal{C}, \mathsf{Set})$ admits all small limit and colimits. Valued at any object of $\mathcal{C}$, the limit (resp. colimit) of a small diagram in $\mathrm{Fun}(\mathcal{C}, \mathsf{Set})$ is the limit (resp. colimit) of the resulting diagram in Set. By considering the opposite category of $\mathcal{C}$, we get the dual statements for the category $\mathsf{PSh}(\mathcal{C})$ instead of $\mathrm{Fun}(\mathcal{C}, \mathsf{Set})$.

Proof. Since Set admits all small limits and colimits, this is derived as a special case from [Sch23, Prop. 2.5.1]. □

Lemma 2.2.11. Let $\mathcal{C}$ be either Sch or Aff and let X, Y and S be schemes. Write h_X for the functor of points of X on $\mathcal{C}^{\mathrm{op}}$. Let $\alpha\colon h_X \to h_S$ and $\beta\colon h_Y \to h_S$ be natural transformations. Then there exist unique morphisms $f\colon X \to S$ and $g\colon Y \to S$ such that the resulting fibered product $X \times_S Y$ represents the fibered product $h_X \times_{h_S} h_Y$ of presheaves on $\mathcal{C}$.

Proof. Applying Proposition 2.2.2 (resp. Proposition 2.2.4) yields the unique morphisms $f\colon X \to S$ and $g\colon Y \to S$ with $\alpha = h(f)$ and $\beta = h(g)$. Since for any object T of $\mathcal{C}$ the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$ preserves limits (see [Rie16, Thm. 3.4.6]), the claim follows from Lemma 2.2.10. □

Convention 2.2.12. Let X be a scheme. For the rest of the report, we write h_X for the functor of points of X, considered as a functor on $\mathsf{Aff}^{\mathrm{op}}$ rather than $\mathsf{Sch}^{\mathrm{op}}$.

See [EH00, VI.1.1] and [Bej20, Def. 2.9] for the following definitions.

Definition 2.2.13 ((open) subfunctors, open covers)**.** Let $F\colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf.

(i) A pair (G, α) consisting of presheaf $G \in \mathsf{PSh}(\mathsf{Aff})$ and a natural transformation $\alpha\colon G \to F$ is called a *subfunctor of* F if for every affine scheme T the map $\alpha_T\colon G(T) \to F(T)$ is injective.

(ii) A subfunctor G of F is called *open* if for every affine scheme T and morphism $h_T \to F$ there exists a representation (X, φ) of the fibered product $G \times_F h_T$, where X is an open subscheme of T via an open immersion $\iota\colon X \hookrightarrow T$ and $h(\iota) = \mathrm{pr}_2 \circ \varphi$ holds.

(iii) A family of open subfunctors $(F_i)_{i\in I}$ is called an *open cover of* F if for every affine scheme T and morphism $h_T \to F$ the family of open subschemes $(T_i)_{i\in I}$ of T, which consists for every $i \in I$ of a representation T_i of the fibered product $F_i \times_F h_T$, forms an open cover of T.

Here are some basic facts about this new notions.

Lemma 2.2.14. (i) Let $\iota\colon X \hookrightarrow Y$ be an open immersion of schemes. Then the map of functors $h(\iota)\colon h_X \to h_Y$ defines an open subfunctor.

(ii) Let X be a scheme and $(X_i)_{i\in I}$ an open cover of X. Then the family of open subfunctors $(h_{X_i})_{i\in I}$ covers h_X.

Being an open subfunctor and being an open cover are properties, which are stable under base change:

(iii) If $F \in \mathsf{PSh}(\mathsf{Aff})$ is a presheaf and G an open subfunctor of F, then for any map $F' \to F$ of presheaves on Aff, the map $\mathrm{pr}_2\colon G \times_F F' \to F'$ defines an open subfunctor.

(iv) Let $F \in \mathsf{PSh}(\mathsf{Aff})$ be a presheaf with an open cover $(F_i)_{i\in I}$ by open subfunctors and let $F' \to F$ be a map of presheaves. Then the family of open subfunctors $(F_i \times_F F')_{i\in I}$ covers F'.

Proof. Let T be an affine scheme.

(i) The map $h_X(T) \to h_Y(T)$ given by composition with ι is injective, since ι is injective. Hence $h(\iota)$ defines a subfunctor. Given a map $h_T \to h_Y$, by Lemma 2.2.11 the functor $h_X \times_{h_Y} h_T$ is represented by $X \times_Y T$, where $T \to Y$ is the unique morphism induced by Proposition 2.2.4. The canonical map $h_{X\times_Y T} \to h_T$ is the morphism given by the projection $X \times_Y T \to T$. It follows by [GW10, Prop. 4.32] that this is an open immersion, showing (i).

(ii) Suppose that there exists a map $h_T \to h_X$. According to (i), the induced map $h_{X_i} \times_{h_X} h_T \to h_T$ is represented by $X_i \times_X T \to T$. Since $(X_i)_{i\in I}$ is an open cover of X, the statement now follows from [GW10, Cor. 4.19].

(iii) Suppose that there exists a map $h_T \to F'$. We have

$$(G \times_F F') \times_{F'} h_T \cong G \times_F h_T$$

by a unique isomorphism. Using that $G \to F$ is an open subfunctor, we get that the functor $(G \times_F F') \times_{F'} h_T$ is representable by the same open subscheme X of T, which represents $G \times_F h_T$, and that the subscheme inclusion induces the projection onto h_T.

(iv) Suppose that there exists a map $h_T \to F'$. As seen in (iii), for every $i \in I$ the functor $(F_i \times_F F') \times_{F'} h_T$ is represented by the open subscheme $X_i \hookrightarrow T$, which represents the functor $F_i \times_F h_T$. These form an open cover of T, since the family $(F_i)_{i\in I}$ covers F. This proves (iv). □

We now prove the representability criterion from [EH00, Thm. VI-14].

Theorem 2.2.15 (representability criterion)**.** Let $F\colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. Then F is representable in Sch if and only if it satisfies the following conditions:

(1) F is a Zariski sheaf.

(2) F admits a cover by affine representable open subfunctors.

Moreover, a representation in Sch of a presheaf $F \in \mathsf{PSh}(\mathsf{Aff})$, which satisfies the condition (1) and (2), is obtained by gluing affine representations of the open subfunctors covering F along representations of their fibered products.

Proof. "⇒": Let X be a scheme. Let T be any scheme with open cover $\mathcal{U}$. A family of morphisms $(f_U)_{U\in\mathcal{U}} \in \prod_{U\in\mathcal{U}} h_X(U)$ with the property that for all $U, V \in \mathcal{U}$ the identity

$$f_U|_{U\cap V} = f_V|_{U\cap V}$$

holds, glues by [GW10, Prop. 3.5] to a unique morphism $f\colon T \to X$ such that $f|_U = f_U$ for every $U \in \mathcal{U}$. Hence h_X satisfies the sheaf property for all schemes. Especially h_X is a Zariski sheaf on Aff.
Any affine open cover $(X_i)_{i\in I}$ of X gives by Lemma 2.2.14(ii) rise to an open cover $(h_{X_i})_{i\in I}$ of h_X by affine representable open subfunctors. Hence h_X satisfies the conditions (1) and (2).

"⇐": Let $F\colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ be a Zariski sheaf, and without loss of generality assume that F is covered by a family of open subfunctors $(\alpha_i\colon h_{X_i} \to F)$, where X_i is for any $i \in I$ an affine scheme.

Strategy for the remainder of the proof. We proceed in three steps: We first show that the family $(X_i)_{i\in I}$ gives rise to a gluing datum for schemes in the sense of Definition 2.1.3, which provides us with a candidate X for a representative of F in Sch. This is a standard argument, see for instance [GW10, Thm. 8.9]. Since X is in general not affine and F is a priori not defined on Sch, we cannot directly use the Zariski sheaf property of F to obtain the desired isomorphism $h_X \to F$. Instead, we define in a next step this natural transformation "manually" by choosing a suitable affine cover in the domain of a morphism $f\colon T \to X$ such that the sheaf property of F applies. Finally we check that this natural transformation is indeed an isomorphism of presheaves.

Step 1: For $i, j \in I$ consider the open subscheme $X_{i,j} \subseteq X_i$, which represents the functor $h_{X_j} \times_F h_{X_i}$ in Sch. Since $h_{X_j} \times_F h_{X_i} \cong h_{X_i} \times_F h_{X_j}$ by a unique isomorphism, we get a unique isomorphism of schemes

$$\varphi_{j,i}\colon X_{i,j} \xrightarrow{\sim} X_{j,i}$$

by Proposition 2.2.7. It is not hard to verify that this defines a gluing datum of schemes (see [GW10, Thm. 8.9]). We thus obtain a scheme X together with a family of open immersions $(\psi_i\colon X_i \to X)_{i\in I}$ such that for $U_i := \psi_i(X_i)$ for $i \in I$, it holds that $X = \bigcup_{i\in I} U_i$ and such that for all $i, j \in I$ we have

$$\psi_i(X_{i,j}) = U_i \cap U_j \qquad \text{and} \qquad \varphi_{j,i} = (\psi_j^{-1} \circ \psi_i)|_{X_{i,j}},$$

see Lemma 2.1.4.

Step 2: Let T be an affine scheme with a morphism $f\colon T \to X$. For each $i \in I$, we choose an affine open cover $\mathcal{V}_i$ of $f^{-1}(U_i)$. Note that for $i, j \in I$ the intersection of two affine schemes $V \in \mathcal{V}_i$ and $W \in \mathcal{V}_j$ is again affine, since T is affine. Hence these intersections make for an affine open cover of $f^{-1}(U_i \cap U_j)$. Let $i, j \in I$, $V \in \mathcal{V}_i$ and $W \in \mathcal{V}_j$ and set

$$f_V^i := \psi_i^{-1} \circ (f|_V)\colon V \to X_i$$

and

$$f_{V,W}^{i,j} := \psi_i^{-1} \circ (f|_{V\cap W})\colon V \cap W \to X_{i,j}.$$

This yields elements

$$(\alpha_i)_V(f_V^i) \in F(V) \qquad \text{and} \qquad (\alpha_i)_{V\cap W}(f_{V,W}^{i,j}) \in F(V \cap W)$$

such that, by naturality of α_i, we get

$$\begin{aligned}((\alpha_i)_V(f_V^i))|_{V\cap W} &= (\alpha_i)_{V\cap W}(f_V^i|_{V\cap W}) = (\alpha_i)_{V\cap W}(f_{V,W}^{i,j}) = (\alpha_j)_{V\cap W}(\varphi_{j,i}(f_{V,W}^{i,j}))\\ &= (\alpha_j)_{V\cap W}((\psi_j^{-1} \circ \psi_i)|_{X_{i,j}} \circ \psi_i^{-1} \circ (f|_{V\cap W}))\\ &= (\alpha_j)_{V\cap W}(\psi_j^{-1} \circ (f|_{V\cap W})) = (\alpha_j)_{V\cap W}(f_{W,V}^{j,i})\\ &= (\alpha_j)_{V\cap W}(f_W^j|_{V\cap W}) = ((\alpha_j)_W(f_W^j))|_{V\cap W}.\end{aligned}$$

Therefore, the sheaf property of F yields a unique element $\alpha_T(f) \in F(T)$ which satisfies

$$\alpha_T(f)|_V = (\alpha_i)_V(f^i_V)$$

for every $i \in I$ and $V \in \mathcal{V}_i$. Since for each $i \in I$ the map $\alpha_i \colon h_{X_i} \to F$ is objectwise injective and since the sheaf property of F yields the same element in $F(T)$ for refinements of the open cover of T, this construction gives rise to a well-defined map

$$\alpha_T \colon h_X(T) \to F(T).$$

We claim that this map is natural in T. Let $g \colon S \to T$ be a morphism of affine schemes and again let $f \in h_X(T)$. Now suppose that $W \subseteq S$ and $V \subseteq T$ are affine open subsets such that $g(W) \subseteq V$ and $f(V) \subseteq U_i$ for some $i \in I$. Then

$$\begin{aligned}(\alpha_S(f \circ g))|_W &= ((\alpha_i)_W(\psi_i^{-1} \circ (f|_V) \circ (g|_W)) = ((\alpha_i)_W \circ h_{X_i}(g|_W))(\psi_i^{-1} \circ (f|_V)) \\ &= (F(g|_W) \circ (\alpha_i)_V)(\psi_i^{-1} \circ (f|_V)) = F(g|_W)((\alpha_T(f))|_V) \\ &= (F(g)(\alpha_T(f)))|_W.\end{aligned}$$

Since we can cover S by affine open subsets W of this form and F is a Zariski sheaf, this shows naturality.

In the final step we verify that the resulting map of presheaves

$$\alpha \colon h_X \to F$$

is an isomorphism, which finishes the proof.

Step 3: Let T be an affine scheme. We construct an inverse to α_T. Let $a \in F(T)$. By Yoneda, this corresponds uniquely to a natural transformation $h_T \to F$ and we get schemes $(T_i)_{i \in I}$ such that for every $i \in I$ the fibered product $h_{X_i} \times_F h_T$ is represented by T_i and the projection $h_{T_i} \to h_T$ is induced by an open immersion $\iota_i \colon T_i \hookrightarrow T$. By assumption, the family $(\iota_i(T_i))_{i \in I}$ is an open cover of T. Let $i \in I$ and $V \subseteq \iota_i(T_i)$ an open subset. The induced morphism $h_V \to h_{X_i}$ in the commutative diagram

$$\begin{array}{ccc} h_{\iota_i^{-1}(V)} & \xrightarrow{\cong} & h_V \\ \downarrow & & \downarrow \\ h_{T_i} & \longrightarrow & h_T \\ \downarrow & & \downarrow \\ h_{X_i} & \longrightarrow & F \end{array}$$

yields by Yoneda a unique morphism $a^i_V \colon V \to X_i$, which satisfies

$$(\alpha_i)_V(a^i_V) = a|_V.$$

For another affine open subset $W \subseteq \iota_i(T_i)$, the intersection $V \cap W = V \times_T W$ is an affine scheme and we get that

$$(\alpha_i)_{V \cap W}(a^i_V|_{V \cap W}) = (a|_V)|_{V \cap W} = (a|_W)|_{V \cap W} = (\alpha_i)_W(a^i_W|_{V \cap W})$$

holds. This shows $a^i_V|_{V \cap W} = a^i_W|_{V \cap W}$ and we can use [GW10, Prop. 3.5] to glue these morphisms to a unique map $a^i \colon \iota_i(T_i) \to X_i$ such that $a^i|_V = a^i_V$ for every affine open subset $V \subseteq T_i$.

Moreover, for $i, j \in I$ and $V \subseteq \iota_i(T_i) \cap \iota_j(T_j)$ affine open, we have

$$(\alpha_i)_V(a^i_V) = a|_V = (\alpha_j)_V(a^j_V),$$

such that the universal property of the fibered product $h_{X_j}(V) \times_{F(V)} h_{X_i}(V)$ yields a unique $a^{i,j}_V \colon V \to X_{i,j}$, which satisfies

$$(X_{i,j} \hookrightarrow X_i) \circ a^{i,j}_V = a^i_V = a^i|_V.$$

As before, for another affine open subset $W \subseteq \iota_i(T_i) \cap \iota_j(T_j)$ the intersection $V \cap W$ is affine and we have

$$((X_{i,j} \hookrightarrow X_i) \circ a^{i,j}_V)|_{V\cap W} = a^i_V|_{V\cap W} = a^i_W|_{V\cap W} = ((X_{i,j} \hookrightarrow X_i) \circ a^{i,j}_W)|_{V\cap W}.$$

Again, we use [GW10, Prop. 3.5] to glue this datum to a unique morphism of schemes $a^{i,j} \colon \iota_i(T_i) \cap \iota_j(T_j) \to X_{i,j}$. For every affine open $V \subseteq \iota_i(V_i) \cap \iota_j(V_j)$ we get $a^{i,j}|_V = a^{i,j}_V$ and the universal property of the fibered product ensures, that

$$(\varphi_{j,i} \circ a^{i,j})|_V = \varphi_{j,i} \circ a^{i,j}_V = a^{j,i}_V = a^{j,i}|_V.$$

holds. This shows $\varphi_{j,i} \circ a^{i,j} = a^{j,i}$. We summarize that the conditions

$$(a^i)^{-1}(X_{i,j}) = \iota_i(T_i) \cap \iota_j(T_j) \quad \text{and} \quad a^j|_{\iota_i(T_i)\cap\iota_j(T_j)} = \varphi_{j,i} \circ (a^i|_{\iota_i(T_i)\cap\iota_j(T_j)})$$

are satisfied. The mapping properties of glued schemes (see [Stacks, Tag 01JB]) yield a unique morphism $\beta_T(a) \colon T \to X$ with $\beta_T(a)|_{\iota_i(T_i)} = \psi_i \circ a^i$ for every $i \in I$. Moreover, the construction of $\beta_T(a)$ did not depend on the choice of the representatives $(T_i)_{i\in I}$ of $h_{X_i} \times_F h_T$, such that

$$\beta_T \colon F(T) \to h_X(T), \qquad a \mapsto \beta_T(a)$$

is a well-defined map. We claim that β_T is inverse to α_T.
Let $a \in F(T)$ and let $(T_i)_{i\in I}$ be as before. For $i \in I$ and an affine open subset $V \subseteq \iota_i(T_i)$ we get the identity

$$\alpha_T(\beta_T(a))|_V = (\alpha_i)_V(\psi_i^{-1} \circ (\beta_T(a)|_V)) = (\alpha_i)_V(a^i|_V) = (\alpha_i)_V(a^i_V) = a|_V.$$

This implies $\alpha_T \circ \beta_T = \mathrm{id}_{F(T)}$.
Consider now a morphism $f \colon T \to X$. Let $i \in I$ and let T_i be a representation of $h_{X_i} \times_F h_T$ with respect to the by $\alpha_T(f)$ induced morphism $h_T \to F$. Given an affine open subset $V \subseteq f^{-1}(U_i)$, the element $\psi_i^{-1} \circ (f|_V) \in h_{X_i}(V)$ satisfies

$$(\alpha_i)_V(\psi_i^{-1} \circ (f|_V)) = \alpha_V(f|_V) = \alpha_T(f)|_V.$$

Using Yoneda and the universal property of the fiber product, this yields a unique element $t \in h_{T_i}(V)$, which satisfies

$$t \circ \iota_i = (V \hookrightarrow T).$$

This shows that $V \subseteq \iota_i(T_i)$, which allows us to compute

$$(\alpha_i)_V(\psi_i^{-1} \circ ((\beta_T(\alpha_T(f)))|_V)) = \alpha_T(f)|_V = (\alpha_i)_V(\psi_i^{-1} \circ (f|_V)).$$

Since $(\alpha_i)_V$ is injective, we obtain

$$(\beta_T(\alpha_T(f)))|_V = f|_V$$

and finally $\beta_T \circ \alpha_T = \mathrm{id}_{h_X(T)}$. □

Remark 2.2.16. (i) If one is familiar with the statement that the restriction of Zariski sheaves on Sch to Zariski sheaves on Aff is an equivalence of categories (see [Stacks, Tag 020W]), the criterion in [GW10, Thm. 8.9] can be used to derive Theorem 2.2.15 and vice versa.

(ii) The version in [EH00, Thm. VI-14] can be obtained from Theorem 2.2.15 by applying the following criterion for open covers: Let $F\colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. A family $(\alpha_i\colon F_i \to F)_{i\in I}$ of open subfunctors is an open cover of F if and only if for every field K the identity

$$\bigcup_{i\in I} (\alpha_i)_{\mathrm{Spec}(K)}(F_i^*(K)) = F^*(K)$$

holds. For "⇒", let $f \in F^*(K)$ and choose $i \in I$ such that $F_i \times_F h_{\mathrm{Spec}(K)}$ is represented by $\mathrm{Spec}(K)$. Then f is in the image of $(\alpha_i)_{\mathrm{Spec}(K)}$.
The direction "⇐" follows basically from [GW10, Prop. 4.8 and Rem. 4.9].

2.3 Representability of the Grassmannian functor

In commutative algebra, the generalization of finite dimensional vector spaces (of dimension n) from linear algebra are finitely generated projective modules (of constant rank n). This leads to the following definition, of what "the Grassmannians" over an arbitrary commutative ring should be:

Definition 2.3.1 (Grassmannian functor)**.** Let $n, N \in \mathbb{N}$ with $1 \leq n < N$. We call the functor

$$\begin{aligned} g_{n,N}^*\colon \quad \mathsf{CRing} &\to \mathsf{Set}, \\ R &\mapsto \left\{ (P,\varphi) \,\middle|\, \begin{array}{l} P \text{ fin. gen. projective } R\text{-module of rank } n \\ \text{and } \varphi\colon R^N \twoheadrightarrow P \text{ epimorphism} \end{array} \right\} / \sim, \\ (f\colon R \to S) &\mapsto \begin{pmatrix} g_{n,N}(f)\colon g_{n,N}^*(R) \to g_{n,N}^*(S), \\ \qquad [(M,\varphi)] \mapsto [(M \otimes_R S, \varphi \otimes \mathrm{id}_S)] \end{pmatrix}, \end{aligned}$$

where $(P,\varphi) \sim (P',\varphi')$ if and only if there exists an R-linear isomorphism $\mu\colon P \to P'$ such that $\varphi' = \mu \circ \varphi$, the *Grassmannian functor*. Often we also mean

$$g_{n,N} := g_{n,N}^* \circ \Gamma\colon \quad \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$$

by the Grassmannian functor.

Remark 2.3.2. Let $n, N \in \mathbb{N}$ with $1 \leq n < N$ and let R be a ring. A rank n direct summand of R^N is a fin. gen. projective R-submodule of R^N of rank n.
Note that the map

$$g^*_{n,N}(R) \to \{\text{rank } N-n \text{ direct summands of } R^N\}, \qquad [(P,\varphi)] \mapsto \ker(\varphi)$$

is a natural bijection. Hence the Grassmannian functor is defined to describe rank $N-n$, instead of rank n, direct summands of R^N. However, one can show that for a field K, the obvious bijection of sets $G_n(K^N) \xrightarrow{\sim} G_{N-n}(K^N)$, given by mapping a subspace of K^N to its orthogonal complement with respect to the standard bilinear form on K^N, is an isomorphism of varieties (see [Gat24, Prop. 8.21]). Hence considering rank n quotients instead of rank n subspaces also fits the classical picture Motivation 2.1.2.

Theorem 2.3.3 (representability of the Grassmannian functor)**.** Let $n, N \in \mathbb{N}$ with $1 \leq n < N$. The Grassmannian functor $g_{n,N} \colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ is represented by the scheme $\mathrm{Gr}_{\mathbb{Z}}(n,N)$.

Before proving the theorem in the general case, we recall the special case $n = 1$, in which we already know a representation of the Grassmannian functor from a previous talk.

Theorem 2.3.4 (functor of points of projective space)**.** Let $N \in \mathbb{N}_{\geq 1}$.

(i) The Grassmannian functor $g_{1,N+1}$ is represented by the projective N-space $\mathbb{P}^N_{\mathbb{Z}}$.

(ii) Given $i \in \{0, \dots, N\}$, the standard affine open chart $U_i := D_+(x_i) \overset{\iota_i}{\hookrightarrow} \mathbb{P}^n_{\mathbb{Z}}$ represents the functor

$$F_i \colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}, \qquad X \mapsto \{[(P,\varphi)] \in g^*_{1,N+1}(\Gamma(X,\mathcal{O}_X)) \mid \langle \varphi(e_i) \rangle_R = P\}$$

such that the diagram

$$\begin{array}{ccc} h_{\mathbb{P}^N_{\mathbb{Z}}} & \xrightarrow{\sim} & g_{n,N+1} \\ {\scriptstyle h(\iota_i)}\big\uparrow & & \big\uparrow \\ h_{U_i} & \xrightarrow{\sim} & F_i \end{array}$$

commutes.

Proof. For (i), we refer to [EH00, Thm. III-37]. Let $i \in \{0, \dots, N\}$ and let X be an affine scheme. The map

$$\begin{aligned} h_{U_i}(X) &\to F_i(X), \\ f &\mapsto \left[\left(\Gamma(X,\mathcal{O}_X), \begin{pmatrix} \Gamma(X,\mathcal{O}_X)^{N+1} \to \Gamma(X,\mathcal{O}_X), \\ e_j \mapsto f^\#(U_i)(x_j/x_i) \end{pmatrix}\right)\right] \end{aligned}$$

is natural. It follows from the fact that being a module isomorphism is a "prime-local" property, that all finitely generated modules of rank 1, which are generated by a single element, are free. This ensures that the map is bijective. Hence we get the desired isomorphism of functors $h_{U_i} \xrightarrow{\sim} F_i$.
To verify commutativity of the diagram in (ii), we quickly recall the construction of the natural bijection $h_{\mathbb{P}^N_{\mathbb{Z}}}(\operatorname{Spec} R) \xrightarrow{\sim} g^*_{1,N+1}(R)$ from [EH00, Cor. III-42] for a ring R.

Given a map of schemes $f\colon \operatorname{Spec} R \to \mathbb{P}^N_{\mathbb{Z}}$, we pull it back along the inclusion morphisms $\iota_j\colon U_j \hookrightarrow \mathbb{P}^N_{\mathbb{Z}}$ to maps

$$f_j\colon X_j := \operatorname{Spec}(R) \times_{\mathbb{P}^N_{\mathbb{Z}}} U_j \to U_j$$

for every $j \in \{0, \dots, N\}$. Then there exists a unique representative of the image of f_j under above natural isomorphism in $F_j(X_j)$ with $e_j \mapsto 1$. This gives rise to a unique epimorphism $\mathcal{O}_{X_j}^{(N+1)} \to \mathcal{O}_{X_j}$ of $\mathcal{O}_{X_j}$-modules (see [GW10, Eq. (7.4.5)]) such that precomposition with the j-th inclusion is the identity on $\mathcal{O}_{X_j}$. By gluing, we obtain a unique epimorphism $\mathcal{O}_{\operatorname{Spec}(R)}^{(N+1)} \to \mathcal{L}$ of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules, where $\mathcal{L}$ is invertible. Now the equivalence of category described in [GW10, Cor. 7.17] tells us, that this exactly corresponds to an element in $g^*_{1,N+1}(R)$.
Now if $f\colon \operatorname{Spec}(R) \to \mathbb{P}^N_{\mathbb{Z}}$ factors as $f = \iota_i \circ f'$ for a morphism $f'\colon \operatorname{Spec}(R) \to U_i$, the pullback $f_i\colon X_i \to U_i$ is just f' and the induced epimorphism

$$\mathcal{O}_{\operatorname{Spec}(R)}^{(N+1)} = \mathcal{O}_{X_i}^{(N+1)} \to \mathcal{O}_{X_i} = \mathcal{O}_{\operatorname{Spec}(R)}$$

is already the morphism of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules obtained by gluing. By applying the global section functor, we see that the image of f in $g^*_{1,N+1}(R)$ thus agrees with the image of f' in $F_i^*(R)$. □

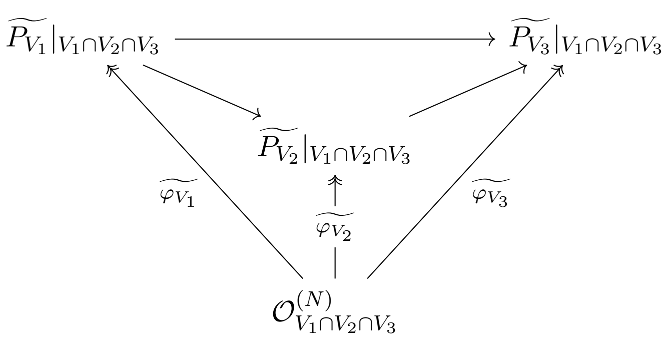
Next we prove representability of the Grassmannian functor by the scheme constructed in Definition 2.1.6. Basic facts about exterior powers of modules can be found for example here [Bou74, Ch. III § 7].

Proof of Theorem 2.3.3. In order to apply Theorem 2.2.15, we verify in a first step that $g_{n,N}$ is a Zariski sheaf. A considerable part of the proof will then be to verify that the two gluing constructions in Lemma 2.1.5 and Theorem 2.2.15 indeed yield up to unique isomorphism the same scheme.

Zariski sheaf property: Let T be an affine scheme with an affine open cover $\mathcal{V}$. Using the equivalence of categories in [GW10, Cor. 7.17] and [GW10, Prop. 7.24], we reformulate the situation in terms of the associated sheaves: The Zariski sheaf property is fulfilled for $g_{n,N}$ if and only if, given for every $V \in \mathcal{V}$ a representative (P_V, φ_V) of a class in $g_{n,N}(V)$ such that

$$\widetilde{P_V}|_{V \cap W} \cong \widetilde{P_W}|_{V \cap W}$$

holds for all $V, W \in \mathcal{V}$ and this isomorphism is compatible with the associated morphisms $\widetilde{\varphi_V}$ and $\widetilde{\varphi_W}$, there exists a unique class in $g_{n,N}(T)$ with a representative (P, φ) such that $\widetilde{P}|_V \cong \widetilde{P_V}$ holds and this isomorphism is compatible with the associated morphisms $\widetilde{\varphi}|_V$ and $\widetilde{\varphi_V}$ for all $V \in \mathcal{V}$. Given $V_1, V_2, V_3 \in \mathcal{V}$, the two lower and the outer triangle commute in the following diagram



This implies that also the upper triangle commutes. Hence [Stacks, Tag 00AL] applies and we obtain by gluing a unique $\mathcal{O}_T$-module $\mathcal{F}$ together with isomorphisms $\mathcal{F}|_V \cong \widetilde{P_V}$ for all $V \in \mathcal{V}$, which are compatible with the isomorphisms on restrictions to intersections. This implies with [Stacks, Tag 04TN] that the family of epimorphisms

$$(\mathcal{O}_V^{(N)} \xrightarrow{\widetilde{\varphi_V}} \widetilde{P_V} \xrightarrow{\sim} \mathcal{F}|_V)_{V\in\mathcal{V}}$$

glues to a unique morphism $\mathcal{O}_T^{(N)} \to \mathcal{F}$. Now without loss of generality we can write $\mathcal{F} = \widetilde{P}$ for a $\Gamma(T, \mathcal{O}_T)$-module P such that the epimorphism $\mathcal{O}_T^{(N)} \to \mathcal{F}$ is given by an epimorphism $\varphi\colon \Gamma(T, \mathcal{O}_T)^N \to P$ of $\Gamma(T, \mathcal{O}_T)$-modules. It remains to check that P is fin. gen. projective of rank n. This follows from the fact that locally on $\mathcal{V}$, the sheaf $\widetilde{P}$ is given by the $\widetilde{P_V}$'s and [GW10, Cor. 7.41].

Application of Theorem 2.2.15: Set $r := \binom{N}{n} - 1$ and consider

$$\mathbb{P}^r_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[x_I \mid I \subseteq \{1, \ldots, N\} \text{ with } \#I = n]).$$

For $I \subseteq \{1, \ldots, N\}$ with $\#I = n$, we set $U_I := D_+(x_I) \subseteq \mathbb{P}^r_{\mathbb{Z}}$. We identify $h_{\mathbb{P}^r_{\mathbb{Z}}}$ with $g_{1,r+1}$ and h_{U_I} with F_I as in Theorem 2.3.4. Let R be a ring and $[(P, \varphi)] \in g^*_{n,N}(R)$. Then by [Bou74, Ch. III § 7.8 Cor. 2] the R-module $\bigwedge^n P$ is projective and the map

$$\textstyle\bigwedge^n \varphi\colon \bigwedge^n R^N \to \bigwedge^n P$$

is an epimorphism by [Bou74, Ch. III § 7.2 Prop. 3]. Especially $\bigwedge^n P$ is finitely generated. Moreover [Bou74, Ch. III § 7.5 Prop. 8 and § 7.8 Thm. 1] yields that $\bigwedge^n P$ is of constant rank $\binom{n}{n} = 1$. This shows $[(\bigwedge^n P, \bigwedge^n \varphi)] \in h^*_{\mathbb{P}^r_{\mathbb{Z}}}(R)$.
By [Bou74, Ch. III § 7.5 Prop. 8] exterior powers commute with scalar extensions. This implies that the induced map

$$g^*_{n,N}(R) \to h^*_{\mathbb{P}^r_{\mathbb{Z}}}(R), \qquad [(P, \varphi)] \mapsto [(\textstyle\bigwedge^n P, \bigwedge^n \varphi)]$$

is natural in R and we obtain a map of functors $g_{n,N} \to h_{\mathbb{P}^r_{\mathbb{Z}}}$. It follows now directly from Lemma 2.2.14(ii) and (iv) that the induced maps of functors

$$g^I_{n,N} := h_{U_I} \times_{h_{\mathbb{P}^r_{\mathbb{Z}}}} g_{n,N} \to g_{n,N}$$

for $I \subseteq \{1, \ldots, N\}$ with $\#I = n$ form an open cover of $g_{n,N}$. We still have to verify that these open subfunctors are representable by affine schemes.
For $I = \{i_1 < \cdots < i_n\} \subseteq \{1, \ldots, N\}$ write $e_I := e_{i_1} \wedge \cdots \wedge e_{i_n} \in \bigwedge^n R^N$ for the I-th standard basis element of $\bigwedge^n R^N$. Then

$$(g^I_{n,N})^*(R) = h^*_{U_i}(R) \times_{h^*_{\mathbb{P}^r_{\mathbb{Z}}}(R)} g^*_{n,N}(R) = \{[(P, \varphi)] \in g^*_{n,N}(R) \mid \langle (\textstyle\bigwedge^n \varphi)(e_I)\rangle = \bigwedge^n P\}$$

holds by Lemma 2.2.10. Hence by [Bou74, Ch. III § 7.2 Prop. 3] any $[(P, \varphi)] \in g^*_{n,N}(R)$ with $\langle \varphi(e_i) \mid i \in I\rangle = P$ lies in $(g^I_{n,N})^*(R)$. Conversely, if $[(P, \varphi)] \in (g^I_{n,N})^*(R)$, it is true by [Bou74, Ch. III § 7.9 Thm. 2] that locally for any prime $\mathfrak{p}$ of R, the family $(\varphi_{\mathfrak{p}}(e_i))_{i\in I}$ is an $R_{\mathfrak{p}}$-basis of the free module $P_{\mathfrak{p}}$. Hence we deduce that P is already a free module with basis $(\varphi(e_i))_{i\in I}$. This shows

$$(g^I_{n,N})^*(R) = \{[(P, \varphi)] \in g^*_{n,N}(R) \mid \langle \varphi(e_i) \mid i \in I\rangle = P\} \qquad (*)$$

and if we consider for a class $[(P,\varphi)] \in (g^I_{n,N})^*(R)$ the unique isomorphism

$$\mu\colon R^I \xrightarrow{\sim} P, \qquad e_i \mapsto \varphi(e_i),$$

this yields a unique representative $(R^n, \mu^{-1}\circ\varphi)$, for which the map $\mu^{-1}\circ\varphi$ is given by a unique matrix $M \in \mathrm{Mat}_{n\times N}(R)$ with $M_I = E_n$. This gives us bijections

$$\begin{aligned}(g^I_{n,N})^*(R) &\cong \{\varphi\colon R^N \to R^I \text{ epimorphism with } e_i \mapsto e_i \text{ for } i \in I\}\\ &\cong \{M \in \mathrm{Mat}_{n\times N}(R) \mid M_I = E_n\} \cong h^*_{W_I}(R), \qquad (**)\end{aligned}$$

where W_I is defined as in Lemma 2.1.5 and the last bijection was discussed in Example 2.2.5. All these bijections are natural in R, so that we get an isomorphism of functors $h_{W_I} \xrightarrow{\sim} g^I_{n,N}$, which shows that the subfunctors given by the composition

$$h_{W_I} \xrightarrow{\sim} g^I_{n,N} \to g_{n,N}$$

is an cover of $g_{n,N}$ by affine representable open subfunctors.
This ensures that we are in the setting of Theorem 2.2.15. Hence there exists a scheme X, which represents $g_{n,N}$ and which is obtained by gluing the W_I's along the induced isomorphisms between representations of the canonically isomorphic fibered products $h_{W_I} \times_{g_{n,N}} h_{W_J}$ and $h_{W_J} \times_{g_{n,N}} h_{W_I}$.

Comparison with Definition 2.1.6: We still have to verify that the so obtained scheme X agrees with our initial definition of the Grassmannian scheme. For $I, J \subseteq \{1,\dots,N\}$ with $\#I = \#J = n$, we again write

$$W_{I,J} := U_J \times_{\mathbb{P}^r_{\mathbb{Z}}} W_I \xhookrightarrow{\iota_{I,J}} W_I$$

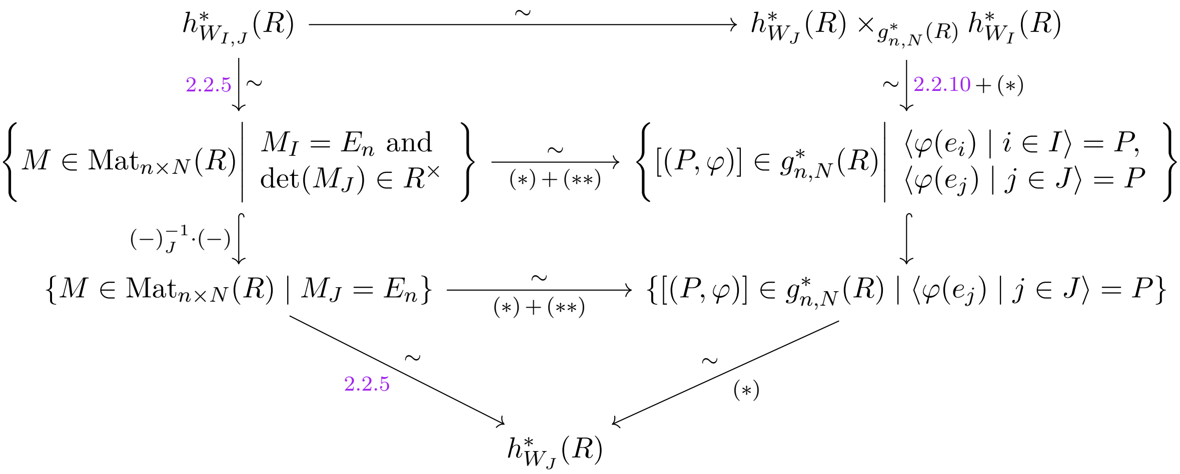
for the affine open subscheme introduced in Lemma 2.1.5 and $\varphi_{J,I}\colon W_{I,J} \xrightarrow{\sim} W_{J,I}$ for the canonical isomorphism of schemes. There exists a canonical isomorphism of functors

$$h_{W_{I,J}} = h_{U_J \times_{\mathbb{P}^r_{\mathbb{Z}}} W_I} \xrightarrow{\sim} h_{U_J} \times_{h_{\mathbb{P}^r_{\mathbb{Z}}}} h_{W_I} \xrightarrow{\sim} (h_{U_J} \times_{h_{\mathbb{P}^r_{\mathbb{Z}}}} g_{n,N}) \times_{g_{n,N}} h_{W_I} \xrightarrow{\sim} h_{W_J} \times_{g_{n,N}} h_{W_I}.$$

In order to see that the gluing datum obtained from Theorem 2.2.15 coincides with the gluing datum defined in Lemma 2.1.5, we have to take a closer look at what maps we obtain by postcomposing this isomorphism with the canonical projections.
Let R be a ring. We have a commutative diagram

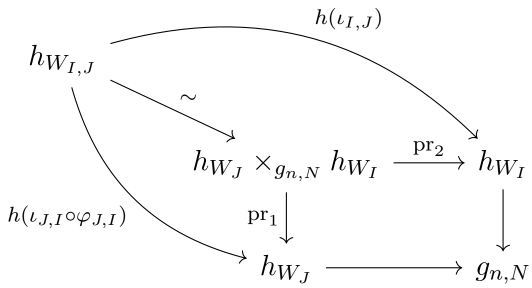
$$\begin{array}{ccc}
h^*_{W_{I,J}}(R) & \xrightarrow{\quad\sim\quad} & h^*_{W_J}(R) \times_{g^*_{n,N}(R)} h^*_{W_I}(R) \\
{\scriptstyle 2.2.5}\downarrow\sim & & \sim\downarrow{\scriptstyle 2.2.10\,+\,(*)} \\
\left\{M \in \mathrm{Mat}_{n\times N}(R) \;\middle|\; \begin{array}{l} M_I = E_n \text{ and} \\ \det(M_J) \in R^\times \end{array}\right\} & \xrightarrow[(*)+(**)]{\sim} & \left\{[(P,\varphi)] \in g^*_{n,N}(R) \;\middle|\; \begin{array}{l} \langle \varphi(e_i) \mid i \in I\rangle = P, \\ \langle \varphi(e_j) \mid j \in J\rangle = P \end{array}\right\} \\
\big\downarrow & & \big\downarrow \\
\{M \in \mathrm{Mat}_{n\times N}(R) \mid M_I = E_n\} & \xrightarrow[(*)+(**)]{\sim} & \{[(P,\varphi)] \in g^*_{n,N}(R) \mid \langle \varphi(e_i) \mid i \in I\rangle = P\} \\
{\scriptstyle 2.2.5}\searrow\sim & & \sim\swarrow{\scriptstyle (*)} \\
& h^*_{W_I}(R) &
\end{array}$$

where the composition of the morphisms on the left hand-side is by Example 2.2.5 the morphism $h^*(\iota_{I,J})\colon h^*_{W_{I,J}}(R) \hookrightarrow h^*_{W_I}(R)$ and the composition of the morphisms on the right hand-side is just the projection on the second component. Similarly, the diagram



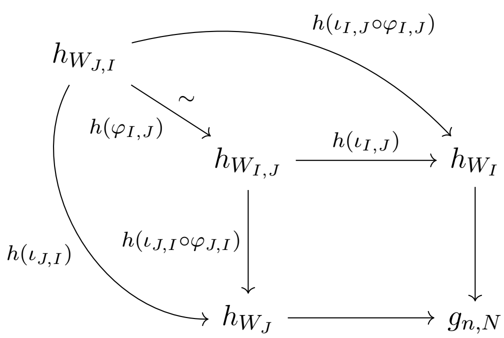
commutes and the composition of the morphisms on the left hand-side is by Example 2.2.5 the morphism $h^*(\iota_{J,I} \circ \varphi_{J,I})\colon h^*_{W_{I,J}}(R) \hookrightarrow h^*_{W_J}(R)$ and the composition of the morphisms on the right hand-side is just projection on the first component.

Combining these two results, we conclude that the diagram



commutes.

This means that $W_{I,J}$ represents the fibered product $h_{W_J} \times_{g_{n,N}} h_{W_I}$ in the sense of Definition 2.2.13(ii). Moreover, the diagram



commutes, which shows that the unique isomorphism from $h_{W_{J,I}}$ to $h_{W_{I,J}}$ induced by the universal property of the fibered product, is exactly given by the map $h(\varphi_{I,J})$.

In view of Theorem 2.2.15, this suffices to conclude that X agrees up to unique isomorphism with the Grassmannian scheme $\mathrm{Gr}_{n,N}(\mathbb{Z})$ from Definition 2.1.6. □

Remark 2.3.5 (non-affine Grassmannian functor)**.** Let $N > n \geq 1$ be natural numbers. The definition of the Grassmannian functor $g_{n,N}$ in Definition 2.1.6 has an obvious generalization to non-affine schemes as well: Since for any ring R, rank n projective R-modules correspond precisely to rank n locally free $\mathcal{O}_{\mathrm{Spec}(R)}$-modules, we may define for a scheme X, the Grassmannian functor of X as

$$g_{n,N}(X) := \left\{ (\mathcal{Q}, \varphi) \;\middle|\; \begin{array}{l} \mathcal{Q} \text{ finite locally free } \mathcal{O}_X\text{-module of rank } n \\ \text{and } \varphi\colon \mathcal{O}_X^{\oplus N} \twoheadrightarrow \mathcal{Q} \text{ epimorphism} \end{array} \right\} / \sim$$

with "$\sim$" as in Definition 2.1.6. In a similar fashion as in Theorem 2.3.3 it is shown here [GW10, Prop. 8.14] that $\mathrm{Gr}_{\mathbb{Z}}(n, N)$ also represents $g_{n,N}$ as a presheaf on schemes.
An advantage of this approach is that we can apply $g_{n,N}$ to $\mathrm{Gr}_{n,N}$ itself and thereby obtain an exceptional element $[\varphi_{\mathrm{univ}}\colon \mathcal{O}^{\oplus N}_{\mathrm{Gr}_{\mathbb{Z}}(n,N)} \twoheadrightarrow \mathcal{Q}_{\mathrm{univ}}]$ in $g_{n,N}(\mathrm{Gr}_{\mathbb{Z}}(n, N))$ which corresponds under Proposition 2.2.2 to the identity on $\mathrm{Gr}_{\mathbb{Z}}(n, N)$. By functoriality of $g_{n,N}$, any morphism of schemes $f\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n, N)$ corresponds now to the pullback $[(f^*\mathcal{Q}_{\mathrm{univ}}, f^*\varphi_{\mathrm{univ}})] \in g_{n,N}(X)$, which explains the name *universal bundle* for this exceptional class. This point of view is convenient for many computations.
For the remainder of the report, however, we again concentrate on the affine case and work with our original definition of $g_{n,N}$ in Definition 2.3.1.

2.4 ∞-Grassmannians

At the beginning of the report, it was already mentioned that in topology, the functor sending a paracompact topological space X to the set of isomorphism classes of real vector bundles on X of a fixed finite rank n is represented up to homotopy by an infinite dimensional Grassmannian manifold, which is defined as the colimit over all finite dimensional Grassmannian manifolds $(G_n(\mathbb{R}^N))_{N \geq n}$. An analogous construction is also feasible in the algebro-geometric setting.

Lemma 2.4.1. Let $n \in \mathbb{N}_{\geq 1}$.

(i) For natural numbers $N, M > n$ with $M \geq N$ and a ring R, the map

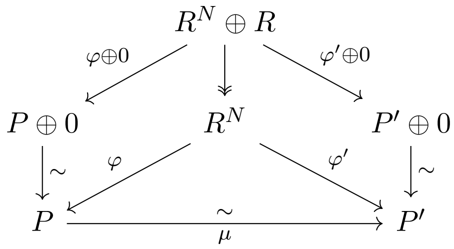
$$\begin{aligned} &(f^n_{N,M})^*_R\colon g^*_{n,N}(R) \to g^*_{n,M}(R), \\ &\qquad [(P, \varphi)] \mapsto [(P, R^M \xrightarrow{\sim} R^N \oplus R^{M-N} \xrightarrow{\varphi \oplus 0} P \oplus 0 \xrightarrow{\sim} P)] \end{aligned}$$

is well-defined and natural in R. Therefore, it gives rise to a map of functors $f^n_{N,M}\colon g_{n,N} \to g_{n,M}$.

(ii) The family $((g_{n,N})_{N>n}, (f^n_{N,M})_{M \geq N > n})$ is a direct system.

Proof. Let $N > n$. For $N = M$, the definition yields the identity on $g_{n,N}$. Let R be a ring. Suppose $M = N + 1$. For well-definedness, let (P, φ) and (P', φ') be two representatives of the same class in $g^*_{n,N}(R)$ and let $\mu\colon P \to P'$ be an isomorphism of R-modules with $\varphi' = \mu \circ \varphi$.

Then the diagram



commutes. This shows

$$(P, R^{N+1} \xrightarrow{\varphi\oplus 0} P) \sim (P', R^{N+1} \xrightarrow{\varphi'\oplus 0} P').$$

Moreover, since direct sums commute with scalar extensions, it is clear that the map $(f^n_{N,N+1})^*_R \colon g^*_{n,N}(R) \to g^*_{n,N+1}(R)$ is natural in R. For arbitrary $M > N$ we can write

$$(f^n_{N,M})^*_R = (f^n_{M-1,M})^*_R \circ \cdots \circ (f^n_{N,N+1})^*_R$$

such that the statement follows from the special case $M = N + 1$. This shows (i) and yields that for $N \leq M \leq K$ it is true that

$$f^n_{M,K} \circ f^n_{N,M} = f^n_{N,K},$$

which implies also (ii). □

Definition 2.4.2 (∞-Grassmannians)**.** Let $n \in \mathbb{N}_{\geq 1}$. In view of Lemma 2.2.10, we define the ∞-*Grassmannian* as the filtered colimit

$$\mathrm{Gr}_n := \mathrm{colim}_{N>n}\, g_{n,N}$$

in $\mathsf{PSh}(\mathsf{Aff})$ with respect to the direct system $((g_{n,N})_{N>n}, (f^n_{N,M})_{M\geq N>n})$, which was constructed in Lemma 2.4.1.

Remark 2.4.3 (∞-Grassmannians as ind-schemes)**.** Following [Ric19, Def. 1.1], we call a filtered colimit of in Sch representable presheaves on affine schemes, for which the transition maps are given by closed immersions, an *ind-scheme*.

The transition map $f^n_{N,N+1} \colon g_{n,N} \to g_{n,N+1}$ is by Proposition 2.2.4 and Theorem 2.3.3 represented by a unique morphism of schemes

$$t^n_{N,N+1} \colon \mathrm{Gr}_{\mathbb{Z}}(n, N) \to \mathrm{Gr}_{\mathbb{Z}}(n, N+1).$$

We can check on the standard affine open cover of $\mathrm{Gr}_{\mathbb{Z}}(n, N+1)$ that $t^n_{N,N+1}$ is a closed immersion:
Let R be a ring. For $J \subseteq \{1, \dots, N\}$ with $\#J = n$, denote by $W^N_J \subseteq \mathrm{Gr}_{\mathbb{Z}}(n, N)$ the affine open subscheme from Lemma 2.1.5. Recalling the proof of Theorem 2.3.3 (more precicely the equations $(*)$ and $(**)$), the open subfunctor $h^*_{W^N_J} \subseteq g^*_{n,N}$ is given by

$$h^*_{W^N_J}(R) \cong \{[(P, \varphi)] \in g^*_{n,N}(R) \mid \langle \varphi(e_j) \mid j \in J\rangle = P\},$$

where $(e_1, \dots, e_N)$ is the standard basis of R^N.

Analogously, for $I \subseteq \{1, \ldots, N+1\}$ with $\#I = n$, we write W_I^{N+1} for the respective affine open subscheme of $\mathrm{Gr}_{\mathbb{Z}}(n, N+1)$ and get

$$h^*_{W_I^{N+1}}(R) \cong \{[(P, \varphi)] \in g^*_{n,N+1}(R) \mid \langle \varphi(e_i) \mid i \in I \rangle = P\},$$

where, by abuse of notation, we denote with e_i the i-th standard basis vector in R^{N+1}. It suffices to check that the restriction of $t^n_{N,N+1}$ to the preimage of W_I^{N+1} is a closed immersion. Therefore, one distinguishes two cases:
Suppose $N+1 \in I$. Let $[(P, \varphi)] \in \mathrm{im}((f^n_{N,N+1})^*_R)$. By construction we have $\varphi(e_{N+1}) = 0$. Hence if we assume

$$P = \langle \varphi(e_i) \mid i \in I \rangle = \langle \varphi(e_i) \mid i \in I \setminus \{N+1\} \rangle,$$

we get that P is finitely generated projective of rank n and admits at the same time $\#(I \setminus \{N+1\}) = n-1$ generators. This is a contradiction, which implies

$$\mathrm{im}((f^n_{N,N+1})^*_R) \cap h^*_{W_I^{N+1}}(R) = \emptyset$$

and equivalently $(t^n_{N,N+1})^{-1}(W_I^{N+1}) = \emptyset$. Hence $(t^n_{N,N+1})|_{\emptyset} = \emptyset$ is trivially a closed immersion.
Now suppose that $N+1 \notin I$, i.e. that I is contained in $\{1, \ldots, N\}$. Then it is obvious by construction that

$$((f^n_{N,N+1})^*_R)^{-1}(h^*_{W_I^{N+1}}(R)) = h^*_{W_I^N}(R)$$

holds such that $t^n_{N,N+1}$ (co)restricts to a morphism

$$t\colon W_I^N \to W_I^{N+1}.$$

Consider the diagram

$$\begin{array}{ccc}
\{M \in \mathrm{Mat}_{n\times N}(R) \mid M_I = E_n\} & \xrightarrow{\text{"}M \mapsto (M \mid 0)\text{"}} & \{M \in \mathrm{Mat}_{n\times(N+1)}(R) \mid M_I = E_n\} \\
\downarrow \wr & & \downarrow \wr \\
h^*_{W_I^N}(R) & \xrightarrow{h^*(t)_R} & h^*_{W_I^{N+1}}(R) \\
\downarrow \wr & & \downarrow \wr \\
\{[(P,\varphi)] \in g^*_{n,N}(R) \mid \langle \varphi(e_i) \mid i \in I \rangle = P\} & \xrightarrow{(f^n_{N,N+1})^*_R} & \{[(P,\varphi)] \in g^*_{n,N+1}(R) \mid \langle \varphi(e_i) \mid i \in I \rangle = P\}
\end{array}$$

with the upper map given by adding a zero column at the end of the matrix and the other arrows as in the proof of Theorem 2.3.3. By construction of $(f^n_{N,N+1})^*_R$ it is clear that the diagram commutes. Moreover, the map "$M \mapsto (M \mid 0)$" in the diagram clearly factors bijectively through the inclusion

$$\left\{ M \in \mathrm{Mat}_{n\times(N+1)}(R) \;\middle|\; \begin{array}{l} M_I = E_n \quad \text{and} \\ (M_{i,N+1})_{1\le i\le n} = 0 \end{array} \right\} \hookrightarrow \{M \in \mathrm{Mat}_{n\times(N+1)}(R) \mid M_I = E_n\}.$$

This map is represented by the inclusion of the closed subscheme

$$Z := W_I^{N+1} \cap V(x_{i,N+1} \mid 1 \le i \le n) \hookrightarrow W_I^{N+1} \subseteq \mathbb{A}_{\mathbb{Z}}^{n(N+1)},$$

where we identify $\mathbb{A}_{\mathbb{Z}}^{n(N+1)}$ with $\operatorname{Spec} \mathbb{Z}[x_{i,j} \mid 1 \le i \le n, 1 \le j \le N+1]$ as in Lemma 2.1.5. This yields by Proposition 2.2.4 that $t\colon W_I^N \to W_I^{N+1}$ factors isomorphically through the closed subscheme $Z \subseteq W_I^{N+1}$, proving that t is a closed immersion. Hence Gr_n is an ind-scheme.

In the more general setting of Remark 2.3.5, one could alternatively argue as follows: Write $\iota_{N+1}\colon \mathcal{O}_{\mathrm{Gr}_{\mathbb{Z}}(n,N+1)} \hookrightarrow \mathcal{O}_{\mathrm{Gr}_{\mathbb{Z}}(n,N+1)}^{\oplus(N+1)}$ for the inclusion into the last component. Then for any scheme X, considering the obvious analog of the map $(f_{N,N+1})_X$ in the non-affine case, we easily check that there exists a commutative diagram of natural maps

$$\begin{array}{ccc} g_{n,N}(X) & \xrightarrow{\sim} & \{g \in \mathrm{Hom}_{\mathsf{Sch}}(X, \mathrm{Gr}_{\mathbb{Z}}(n, N+1)) \mid g^*(\varphi_{\mathrm{univ}} \circ \iota_{N+1}) = 0\} \\ {\scriptstyle (f_{N,N+1})_X}\downarrow & & \downarrow \\ g_{n,N+1}(X) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathsf{Sch}}(X, \mathrm{Gr}_{\mathbb{Z}}(n, N+1)) \end{array}$$

Since the universal bundle is a finite locally free $\mathcal{O}_{\mathrm{Gr}_{\mathbb{Z}}(n,N+1)}$-module, it is a general fact from algebraic geometry (see [GW10, Prop. 8.4]) that the vertical map on the right hand-side in the diagram is as a natural transformation represented by the inclusion of a closed subscheme.

3 Naive $\mathbb{A}^1$-homotopical Classification of Algebraic Vector Bundles

3.1 Naive $\mathbb{A}^1$-homotopy revisited

In order to classify algebraic vector bundles up to naive $\mathbb{A}^1$-homotopy in terms of maps into the ∞-Grassmannian from the previous subsection, we have to make sense of what we mean by naive $\mathbb{A}^1$-homotopies between morphisms, where the target is a presheaf on affine schemes. Therefore, we extend the notions in [Aso19, Def. 2.2.2.1 and Def. 2.2.2.4], which are already known from the earlier talks, by translating them into the language of functors of points.

Notation 3.1.1. Let k be a field and $X = \operatorname{Spec}(R)$ an affine k-scheme. For any $r \in R$, there exists a unique evaluation map $\mathrm{ev}_r \colon R[x] \to R$ with $x \mapsto r$. We write ε_r for the composition

$$X = \operatorname{Spec}(R) \xrightarrow{\operatorname{Spec}(\mathrm{ev}_r)} \mathbb{A}^1_R \xrightarrow{\sim} X \times_k \mathbb{A}^1_k.$$

Definition 3.1.2 (Naive $\mathbb{A}^1$-homotopies)**.** Let k be a field, X an affine k-scheme and $F \colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ a presheaf on affine schemes. Let $f, g \colon h_X \to F$ be two maps of functors. A *naive $\mathbb{A}^1$-homotopy from f to g* is a natural transformation $H \colon h_{X \times_k \mathbb{A}^1_k} \to F$, which satisfies

$$H \circ h(\varepsilon_0) = f \qquad \text{and} \qquad H \circ h(\varepsilon_1) = g.$$

We also write $H \colon f \Rightarrow g$, if H is a naive $\mathbb{A}^1$-homotopy from f to g.

Remark 3.1.3. Let k be a field, X an affine k-scheme and F a presheaf on affine schemes. Let $f, g \colon h_X \to F$ be maps of functors.

(i) By Proposition 2.2.2 it is clear that in the case $F = h_Y$ for an affine k-scheme Y, a naive $\mathbb{A}^1$-homotopy from f to g in the sense of Definition 3.1.2 is precicely a naive $\mathbb{A}^1$-homotopy in the sense of [Aso19, Def. 2.2.2.1].

(ii) The relation given by "existence of a naive $\mathbb{A}^1$-homotopy from f to g" is reflexive and symmetric. The proof of this is a straighforward adaption of the arguments in [Aso19, p. 27]. However, as demonstrated here [Aso19, Ex. 2.2.2.3], transitivity fails by part (i) already in the case, where F is represented by an affine k-scheme.

Definition 3.1.4 (Naive $\mathbb{A}^1$-homotopy equivalence)**.** Let k be a field, let X be an affine k-scheme and $F \colon \mathsf{Aff}^{\mathrm{op}} \to \mathsf{Set}$ a presheaf on affine schemes. We write "$\sim_N$" for the equivalence relation on $\operatorname{Hom}_{\mathsf{PSh}(\mathsf{Aff})}(h_X, F)$ generated by the relation

$$f \sim g \quad :\Longleftrightarrow \quad \exists \, \text{naive } \mathbb{A}^1 - \text{homotopy from } f \text{ to } g$$

and we write

$$[X, F]_N := \operatorname{Hom}_{\mathsf{PSh}(\mathsf{Aff})}(h_X, F) / \sim_N$$

for the quotient set. Two maps $f, g \colon h_X \to F$ are called *naively $\mathbb{A}^1$-homotopic* if $f \sim_N g$.

Remark 3.1.5. Let k be a field, X an affine k-scheme and F a presheaf on affine schemes. By Remark 3.1.3(ii), two natural transformations $f, g\colon h_X \to F$ are naively $\mathbb{A}^1$-homotopic if and only if there exists an $n \in \mathbb{N}_{\geq 2}$ and morphisms $f_1, \dots, f_n\colon h_X \to F$ such that $f_1 = f$, $f_n = g$ and there exists a naive $\mathbb{A}^1$-homotopy $H_i\colon f_i \Rightarrow f_{i+1}$ for any $i \in \{1, \dots, n-1\}$.

Recall the definition of naive $\mathbb{A}^1$-invariance from [Aso19, Def. 2.2.1.1]:

Definition 3.1.6. Let k be a field, $\mathcal{C}$ a category and $\mathcal{F}\colon \mathsf{Aff}_k^{\mathrm{op}} \to \mathcal{C}$ a functor. Then $\mathcal{F}$ is called *naively $\mathbb{A}^1$-invariant* if $\mathcal{F}(\mathrm{pr}_1)\colon \mathcal{F}(X) \to \mathcal{F}(X \times_k \mathbb{A}^1_k)$ is an isomorphism.

Let us establish the following essential result.

Proposition 3.1.7. Let k be a field and $F\colon \mathsf{Aff}_k^{\mathrm{op}} \to \mathsf{Set}$ a presheaf.

(i) The functor
$$[-, F]_N\colon \quad \mathsf{Aff}_k^{\mathrm{op}} \to \mathsf{Set}, \qquad X \mapsto [X, F]_N,$$
which is given on morphisms by pullback, is naively $\mathbb{A}^1$-invariant.

(ii) The natural transformation
$$\eta\colon \quad \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Aff}_k)}(h_{(-)}, F) \ \to\ [-, F]_N,$$
which, valued at an affine k-scheme X, sends a map of functors $h_X \to F$ to its naive $\mathbb{A}^1$-homotopy class, has the following property: Given a naively $\mathbb{A}^1$-invariant functor $\mathcal{F}\colon \mathsf{Aff}_k^{\mathrm{op}} \to \mathsf{Set}$ together with a natural transformation
$$\theta\colon \quad \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Aff}_k)}(h_{(-)}, F) \ \to\ \mathcal{F},$$
there exists a unique natural transformation $\overline{\theta}\colon [-, F]_N \to \mathcal{F}$ with $\overline{\theta} \circ \eta = \theta$.

Proof. Note that by Proposition 2.2.2 for any affine k-scheme X, natural transformations from h_X to F form a set. Especially we get that $[X, F]_N$ is a set.

(i) First of all, note that $[-, F]_N$ is indeed a functor: Let X and Y be affine k-schemes, $f, f'\colon X \to Y$ morphisms of schemes and $g, g'\colon h_Y \to F$ maps of functors. Suppose that $f \sim_N f'$ and $g \sim_N g'$. It suffices to check that $g \circ h(f)$ and $g' \circ h(f')$ are naively $\mathbb{A}^1$-homotopic.
The case where F is represented by an affine k-scheme, is already known from [Aso19, Lem. 2.2.2.2]. First of all suppose, that there exist naive $\mathbb{A}^1$-homotopies $H\colon f \Rightarrow f'$ and $H'\colon g \Rightarrow g'$. By the universal property of the fibered product, we get a map $H \times \mathrm{pr}_2\colon X \times_k \mathbb{A}^1_k \to Y \times_k \mathbb{A}^1_k$. Set
$$K := H' \circ h(H \times \mathrm{pr}_2).$$
By construction, the diagram
$$\begin{array}{ccc} h_X & \xrightarrow{\ h(f)\ } & h_Y \\ \big\downarrow {\scriptstyle h(\varepsilon_0^X)} & \nearrow {\scriptstyle h(H)} & \big\downarrow {\scriptstyle h(\varepsilon_0^Y)} \\ h_{X \times_k \mathbb{A}^1_k} & \xrightarrow{\ h(H \times \mathrm{pr}_2)\ } & h_{Y \times_k \mathbb{A}^1_k} \end{array}$$
commutes.

This implies that

$$K \circ h(\varepsilon_0^X) = H' \circ h(H \times \mathrm{pr}_2) \circ h(\varepsilon_0^X) = H' \circ h(\varepsilon_0^Y) \circ h(f) = g \circ h(f).$$

An analogous argument shows that

$$K \circ h(\varepsilon_1^X) = g' \circ h(f').$$

For the general case, we choose finite sequences of naive $\mathbb{A}^1$-homotopies

$$f = f_1 \Rightarrow \cdots \Rightarrow f_n = f' \qquad \text{and} \qquad g = g_1 \Rightarrow \cdots \Rightarrow g_m = g'.$$

From the special case, we obtain naive $\mathbb{A}^1$-homotopies

$$g \circ h(f) = g \circ h(f_1) \Rightarrow \cdots \Rightarrow g \circ h(f_n) = g_1 \circ h(f') \Rightarrow \cdots \Rightarrow g_m \circ h(f') = g' \circ h(f').$$

Hence $[-, F]_N$ is a functor.

The proof of naive $\mathbb{A}^1$-invariance is essentially the same as [Aso19, Lemma 2.2.3.2]. Let $X = \mathrm{Spec}(R)$ be an affine k-scheme. For any $r \in R$, the map $\varepsilon_r \colon X \to X \times_k \mathbb{A}_k^1$ is a splitting of the canonical projection onto the first component. Hence by functoriality the map $\mathrm{pr}_1^* \colon [X, F]_N \to [X \times_k \mathbb{A}_k^1, F]_N$ is injective. For the proof of surjectivity, suppose that $f \colon h_{X \times_k \mathbb{A}_k^1} \to F$ is any morphism of functors. We show that f is naively $\mathbb{A}^1$-homotopic to $f \circ h(\varepsilon_0 \circ \mathrm{pr}_1)$. In the case of $(X \times_k \mathbb{A}_k^1) \times_k \mathbb{A}_k^1$, we write

$$\varepsilon_i' \colon \quad X \times_k \mathbb{A}_k^1 \xrightarrow{\sim} \mathbb{A}_R^1 \to \mathbb{A}_R^1 \times_k \mathbb{A}_k^1 \xrightarrow{\sim} (X \times_k \mathbb{A}_k^1) \times_k \mathbb{A}_k^1$$

for the respective maps for $i \in \{0, 1\}$. Denote by $m \colon \mathbb{A}_k^1 \times \mathbb{A}_k^1 \to \mathbb{A}_k^1$ the morphism associated to the "multiplication map"

$$k[t] \to k[x] \otimes_k k[y], \qquad t \mapsto x \otimes y.$$

The claim is that the composition of natural transformations

$$H: \quad h_{(X \times_k \mathbb{A}_k^1) \times_k \mathbb{A}_k^1} \xrightarrow{h(\mathrm{id}_X \times m)} h_{X \times_k \mathbb{A}_k^1} \xrightarrow{f} F$$

defines a naive $\mathbb{A}^1$-homotopy from $f \circ h(\varepsilon_0 \circ \mathrm{pr}_1)$ to f. This follows from

$$\begin{aligned} H \circ h(\varepsilon_0') &= f \circ h((\mathrm{id}_X \times m) \circ \varepsilon_0') \\ &= f \circ h^* \left(k[t] \xrightarrow{t \mapsto xy} k[x, y] \xrightarrow{(x,y) \mapsto (t,0)} k[t] \right) \\ &= f \circ h^* \left(k[t] \xrightarrow{t \mapsto 0} k[t] \right) = f \circ h^* \left(k[t] \xrightarrow{t \mapsto 0} k \hookrightarrow k[t] \right) \\ &= f \circ h(\varepsilon_0 \circ \mathrm{pr}_1) \end{aligned}$$

and

$$\begin{aligned} H \circ h(\varepsilon_1') &= f \circ h((\mathrm{id}_X \times m) \circ \varepsilon_1') \\ &= f \circ h^* \left(k[t] \xrightarrow{t \mapsto xy} k[x, y] \xrightarrow{(x,y) \mapsto (t,1)} k[t] \right) \\ &= f \circ h^* \left(k[t] \xrightarrow{t \mapsto t} k[t] \right) = f. \end{aligned}$$

Since $f \circ h(\varepsilon_0 \circ \mathrm{pr}_1) = \mathrm{pr}_1^*(f \circ h(\varepsilon_0))$, this shows surjectivity.

(ii) It is obvious that η defines a natural transformation, which is surjective valued at any point. Let $\mathcal{F}\colon \mathsf{Aff}_k^{\mathrm{op}} \to \mathsf{Set}$ be a naively $\mathbb{A}^1$-invariant functor and θ as in (ii). If we manage to show that a natural transformation $\overline{\theta}\colon [-,F]_N \to \mathcal{F}$ with $\overline{\theta}\circ\eta = \theta$ exists, it is clear by surjectivity of η that it has to be unique.
Let $X = \mathrm{Spec}(R) \in \mathsf{Aff}_k$. Suppose $f, g\colon h_X \to F$ are maps of functors, which are connected by a naive $\mathbb{A}^1$-homotopy $H\colon f \Rightarrow g$. Since $\mathcal{F}$ is naively $\mathbb{A}^1$-homotopy invariant, we get that $\mathrm{pr}_1\colon X \times_k \mathbb{A}^1_k \to X$ is sent under $\mathcal{F}$ to a bijection. This map of schemes is induced by the inclusion $\iota\colon R \hookrightarrow R[x]$, which satisfies for any $r \in R$ that $\mathrm{ev}_r \circ \iota = \mathrm{id}_R$. By functoriality, this implies

$$\mathcal{F}(\varepsilon_r) = \mathcal{F}(\mathrm{pr}_1)^{-1}$$

for all $r \in R$. Hence the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Aff})}(h_X, F) & \xrightarrow{\theta_X} & \mathcal{F}(X) \\ {\scriptstyle h(\varepsilon_0)^*}\uparrow\ \uparrow{\scriptstyle h(\varepsilon_1)^*} & & \uparrow{\scriptstyle \mathcal{F}(\mathrm{pr}_1)^{-1}} \\ \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Aff})}(h_{\mathbb{A}^1_X}, F) & \xrightarrow{\theta_{\mathbb{A}^1_X}} & \mathcal{F}(\mathbb{A}^1_X) \end{array}$$

commutes. This means that

$$\begin{aligned} \theta_X(f) &= \theta_X(H \circ h(\varepsilon_0)) = (\theta_X \circ h(\varepsilon_0)^*)(H) \\ &= (\mathcal{F}(\mathrm{pr}_1)^{-1} \circ \theta_{\mathbb{A}^1_X})(H) = (\theta_X \circ h(\varepsilon_1)^*)(H) \\ &= \theta_X(H \circ h(\varepsilon_1)) = \theta_X(g). \end{aligned}$$

By induction, we get that $\theta_X(f) = \theta_X(g)$ also holds for naively $\mathbb{A}^1$-homotopic $f, g\colon h_X \to F$. Hence

$$\overline{\theta}_X\colon [X,F]_N \to \mathcal{F}(X), \qquad [f] \mapsto \theta_X(f)$$

is well-defined and yields the natural transformation with the desired properties. □

3.2 The classification theorem

For the final subsection of the report, we recall the definition of algebraic vector bundles of a finite fixed rank.

Definition 3.2.1 (Algebraic vector bundles)**.** Let $n \in \mathbb{N}$. We call the functor

$$\begin{aligned} \mathrm{Vect}_n\colon \quad \mathsf{Aff}^{\mathrm{op}} &\to \mathsf{Set}, \\ \mathrm{Spec}(R) &\mapsto \left\{ P \in \mathsf{Mod}(R) \;\middle|\; \begin{array}{l} P \text{ fin. gen. projective} \\ R\text{-module of rank } n \end{array} \right\} / \cong \\ (\mathrm{Spec}(S) \xrightarrow{\mathrm{Spec}(\mu)} \mathrm{Spec}(R)) &\mapsto \left(\begin{array}{r} \mu_*\colon \mathrm{Vect}_n(\mathrm{Spec}(R)) \to \mathrm{Vect}_n(\mathrm{Spec}(S)), \\ {[P]} \mapsto [P \otimes_\mu S] \end{array} \right) \end{aligned}$$

the *(rank n) algebraic vector bundle functor.*

In several previous talks we layed the groundwork for and proved the following difficult theorem.

Theorem 3.2.2 (Quillen-Suslin-Lindel Theorem)**.** Let k be a field and $n \in \mathbb{N}_{\geq 1}$. The restricted algebraic vector bundle functor

$$\mathrm{Vect}_n\colon \quad \mathsf{AffSm}_k^{\mathrm{op}} \to \mathsf{Set}$$

is naively $\mathbb{A}^1$-invariant.

Proof. Any smooth affine scheme over k is regular and of finite type - and therefore especially essentially of finite type - over k. Hence the statement follows directly from [Aso19, Thm. 8.4.3.1]. □

Remark 3.2.3. Let $n \in \mathbb{N}_{\geq 1}$ and X an affine scheme. For $M \geq N > n$, recall from Lemma 2.4.1 the map $f^n_{N,M}$ and from Remark 2.4.3 the associated map $t^n_{N,M}$. Since our goal is to classify algebraic vector bundles in terms of maps into the ∞-Grassmannians, it is convenient to once unpack the definition of this Hom-Sets by means of Proposition 2.2.2 and Lemma 2.2.10:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{PSh(Aff)}}(h_X, \mathrm{Gr}_n) &\cong \mathrm{Gr}_n(X) = \mathrm{colim}_{N>n}\, g_{n,N}(X) \\ &= \left(\coprod_{N>n} g_{n,N}(X)\right) \Big/ \left(\begin{array}{l} (N,x) \sim (M,y) \\ :\Longleftrightarrow \left(f^n_{N,\max(N,M)}\right)_X (x) = \left(f^n_{M,\max(N,M)}\right)_X (y) \end{array} \right),\end{aligned}$$

where we use in the last step that the transition maps are injective. The representability of the finite dimensional Grassmannian functor Theorem 2.3.3 allows us to further make the identification

$$\mathrm{Hom}_{\mathsf{PSh(Aff)}}(h_X, \mathrm{Gr}_n) \cong \frac{\coprod_{N>n} \mathrm{Hom}_{\mathsf{PSh(Aff)}}(X, \mathrm{Gr}_{\mathbb{Z}}(n,N))}{(N,f) \sim (M,g) \ \ :\Longleftrightarrow \ \left(t^n_{N,\max(N,M)} \circ f = t^n_{M,\max(N,M)} \circ g\right)}.$$

The bijections in both displays are natural in X. For the rest of the report we will frequently switch between these explicit descriptions.

The next two lemmata help us to better understand when two maps $f, g\colon h_X \to \mathrm{Gr}_n$ are naively $\mathbb{A}^1$-homotopic.

Lemma 3.2.4. Let k be a field and $n \in \mathbb{N}_{\geq 1}$. Let $X = \mathrm{Spec}(R)$ be an affine k-scheme and $f, g\colon h_X \to \mathrm{Gr}_n$ be maps of functors. Suppose that f is given as the equivalence class of the map $f_r\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n,r)$ and g is given as the equivalence class of the map $g_s\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n,s)$ for natural numbers $r, s > n$. Then f and g are naively $\mathbb{A}^1$-homotopic if and only if there exists a natural number $N \geq \max(r,s)$ such that $t^n_{r,N} \circ f_r$ and $t^n_{s,N} \circ g_s$ are naively $\mathbb{A}^1$-homotopic.

Proof. Consider any map $H_N\colon \mathbb{A}^1_R \to \mathrm{Gr}_{\mathbb{Z}}(n,N)$ for some $N > n$ and the induced map $H = [H_N]\colon h_{\mathbb{A}^1_R} \to \mathrm{Gr}_n$. For all $r \in R$, we have

$$H \circ h(\varepsilon_r) = h(\varepsilon_r)^*([H_N]) = [\varepsilon_r^*(H_N)] = [H_N \circ \varepsilon_r].$$

This already implies "$\Leftarrow$".

For the opposite direction, note that any naive $\mathbb{A}^1$-homotopy $H\colon f \Rightarrow g$ is of the form $[H_N]$ for some $N > n$ and without loss of generality we may assume that $N \geq \max(r, s)$. Then by construction, the identity

$$[t^n_{r,N} \circ f_r] = f = H \circ h(\varepsilon_0) = [H_N \circ \varepsilon_0]$$

means that

$$t^n_{r,N} \circ f_r = H_N \circ \varepsilon_0.$$

Analogously also

$$t^n_{s,N} \circ g_s = H_N \circ \varepsilon_1$$

holds. Thus H_N is a naive $\mathbb{A}^1$-homotopy from $t^n_{r,N} \circ f_r$ to $t^n_{s,N} \circ g_s$. We use this to show the general case: For a given chain of naive $\mathbb{A}^1$-homotopies

$$f = f^{(1)} \xRightarrow{H^{(1)}} f^{(2)} \Longrightarrow \ldots \xRightarrow{H^{(k-1)}} f^{(k)} = g,$$

where for all $i \in \{1, \ldots, k\}$ the map $f^{(i)}\colon h_X \to \mathrm{Gr}_n$ is given by $f^{(i)}_{r_i}\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n, r_i)$ for an $r_i > n$, we obtain for each $i \in \{1, \ldots, k-1\}$ an $N_i \geq \max(r_i, r_{i+1})$ such that the naive $\mathbb{A}^1$-homotopy $H^{(i)}\colon h_{\mathbb{A}^1_R} \to \mathrm{Gr}_n$ is given by a naive $\mathbb{A}^1$-homotopy

$$H^{(i)}_{N_i}\colon \quad t^n_{r_i,N_i} \circ f^{(i)}_{r_i} \implies t^n_{r_{i+1},N_i} \circ f^{(i+1)}_{r_{i+1}}.$$

Set $N := \max\{N_i \mid 1 \leq i \leq k-1\}$. For every $i \in \{1, \ldots, k-1\}$, we obtain a naive $\mathbb{A}^1$-homotopy

$$t^n_{N_i,N} \circ H^{(i)}_{N_i}\colon \quad t^n_{r_i,N} \circ f^{(i)}_{r_i} \implies t^n_{r_{i+1},N} \circ f^{(i+1)}_{r_{i+1}},$$

which shows "$\Rightarrow$". □

Lemma 3.2.5. Let k be a field and $n \in \mathbb{N}_{\geq 1}$. Let $X = \mathrm{Spec}(R)$ be an affine k-scheme, $N > n$ a natural number and $f, g\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n, N)$ morphisms of schemes such that f is given by $[(P, \varphi)]$ and g is given by $[(P, \psi)]$ in $g_{n,N}(X)$. Let $[(P[t], \Phi)] \in g_{n,N}(\mathbb{A}^1_R)$ and $H\colon X \times_k \mathbb{A}^1_k \to \mathrm{Gr}_{\mathbb{Z}}(n, N)$ the corresponding map. Then H is a naive $\mathbb{A}^1$-homotopy from f to g if and only if the identities

$$[(P, \Phi \otimes_{\mathrm{ev}_0} R)] = [(P, \varphi)] \qquad \text{and} \qquad [(P, \Phi \otimes_{\mathrm{ev}_1} R)] = [(P, \psi)]$$

hold.

Proof. This is a consequence of the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Sch}}(\mathbb{A}^1_X, \mathrm{Gr}_{\mathbb{Z}}(n, N)) & \xrightarrow{\sim} & g_{n,N}(\mathbb{A}^1_X) \\ \downarrow{\scriptstyle \varepsilon_r^*} & & \downarrow{\scriptstyle \varepsilon_r^*} \\ \mathrm{Hom}_{\mathsf{Sch}}(X, \mathrm{Gr}_{\mathbb{Z}}(n, N)) & \xrightarrow{\sim} & g_{n,N}(X) \end{array}$$

for all $r \in R$. Write $\iota\colon R \to R[t]$ for the canonical inclusion. Then $\mathrm{ev}_r \circ \iota = \mathrm{id}_R$ implies that $P[t] \otimes_{\mathrm{ev}_r} R = P$ holds. Hence we get for any $[(P[t], \Phi)] \in g_{n,N}(\mathbb{A}^1_X)$, that

$$\varepsilon_r^*([(P[t], \Phi)]) = [(P, \Phi \otimes_{\mathrm{ev}_r} R)].$$

Thus for $H\colon \mathbb{A}^1_X \to \mathrm{Gr}_{\mathbb{Z}}(n,N)$ given by $[(P[t],\Phi)] \in g_{n,N}(\mathbb{A}^1_X)$, we obtain equivalences

$$\begin{aligned} &H \text{ is a naive } \mathbb{A}^1\text{-homotopy from } f \text{ to } g \\ \iff\quad & \varepsilon_0^*(H) = f \quad \text{and} \quad \varepsilon_1^*(H) = g \\ \iff\quad & [(P, \Phi \otimes_{\mathrm{ev}_0} R)] = [(P,\varphi)] \quad \text{and} \quad [(P, \Phi \otimes_{\mathrm{ev}_1} R)] = [(P,\psi)] \end{aligned}$$

This shows the claim. □

The following corollary serves both as an example of how the previous two lemmata apply together in practice and as an important step in the proof of the classification theorem.

Corollary 3.2.6. Let k be a field and $n \in \mathbb{N}_{\geq 1}$. Let $X = \mathrm{Spec}(R)$ be some affine k-scheme, $s > n$ a natural number and $[(P,\psi)] \in g_{n,s}(X)$. Let $r \in \mathbb{N}$ and set $N := r + s$. The maps $f\colon h_X \to \mathrm{Gr}_n$, given by $[(P, \psi \oplus 0^r)] \in g_{n,N}(X)$, and $g\colon h_X \to \mathrm{Gr}_n$, given by $[(P, 0^r \oplus \psi)] \in g_{n,N}(X)$, are naively $\mathbb{A}^1$-homotopic.

Proof. By assumption, we can consider representation $f_{N+s}\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n, N+s)$ of f and $g_{N+s}\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n, N+s)$ of g. We construct a chain of homotopies from f_{N+s} to g_{N+s}. Define therefore the following epimorphisms of $R[t]$-modules:

$$h_1\colon \quad R[t]^{N+s} \to P[t], \qquad e_i \mapsto \begin{cases} \psi(e_i), & \text{if } 1 \leq i \leq s, \\ 0, & \text{if } s+1 \leq i \leq N, \\ t\psi(e_{i-N}), & \text{if } N+1 \leq i \leq N+s, \end{cases}$$

$$h_2\colon \quad R[t]^{N+s} \to P[t], \qquad e_i \mapsto \begin{cases} (1-t)\psi(e_i), & \text{if } 1 \leq i \leq s, \\ 0, & \text{if } s+1 \leq i \leq N, \\ \psi(e_{i-N}), & \text{if } N+1 \leq i \leq N+s, \end{cases}$$

$$h_3\colon \quad R[t]^{N+s} \to P[t], \qquad e_i \mapsto \begin{cases} 0, & \text{if } 1 \leq i \leq r, \\ t\psi(e_{i-r}), & \text{if } r+1 \leq i \leq N, \\ \psi(e_{i-N}), & \text{if } N+1 \leq i \leq N+s, \end{cases}$$

$$h_4\colon \quad R[t]^{N+s} \to P[t], \qquad e_i \mapsto \begin{cases} 0, & \text{if } 1 \leq i \leq r, \\ \psi(e_{i-r}), & \text{if } r+1 \leq i \leq N, \\ (1-t)\psi(e_{i-N}), & \text{if } N+1 \leq i \leq N+s. \end{cases}$$

It is now easy to check that

$$\begin{aligned} h_1 \otimes_{\mathrm{ev}_0} R &= \psi \oplus 0^r \oplus 0^s, \\ h_i \otimes_{\mathrm{ev}_1} R &= h_{i+1} \otimes_{\mathrm{ev}_0} R \qquad \text{for } 1 \leq i \leq 3 \quad \text{and} \\ h_4 \otimes_{\mathrm{ev}_1} R &= 0^r \oplus \psi \oplus 0^s. \end{aligned}$$

Denote for all $i \in \{1,\dots,4\}$ by H_i the map $\mathbb{A}^1_R \to \mathrm{Gr}_{\mathbb{Z}}(n, N+s)$ corresponding to $[(P[t], h_i)] \in g_{n,N+s}(\mathbb{A}^1_R)$. By Lemma 3.2.5 this defines a chain of naive $\mathbb{A}^1$-homotopies

$$f_{N+s} \overset{H_1}{\Longrightarrow} H_1 \circ \varepsilon_1 \overset{H_2}{\Longrightarrow} H_2 \circ \varepsilon_1 \overset{H_3}{\Longrightarrow} H_3 \circ \varepsilon_1 \overset{H_4}{\Longrightarrow} g_{N+s}.$$

Hence by Lemma 3.2.4, f and g are naively $\mathbb{A}^1$-homotopic. □

Theorem 3.2.7 (Classification Theorem)**.** Let k be a field and $n \in \mathbb{N}_{\geq 1}$. There exists a natural transformation

$$\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Aff})}(h_{(-)}, \mathrm{Gr}_n) \;\to\; \mathrm{Vect}_n,$$

which induces, restricted to AffSm_k, an isomorphism of presheaves

$$[-, \mathrm{Gr}_n]_N \;\to\; \mathrm{Vect}_n\,.$$

Proof. Let X be an affine scheme. From the description of morphisms into Gr_n in Remark 3.2.3, we obtain an assignment

$$\beta_X\colon \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Aff})}(h_X, \mathrm{Gr}_n) \to \mathrm{Vect}_n(X),$$

which sends a morphisms, given by a class $[(P,\varphi)] \in g_{n,N}(X)$ for some $N > n$, to the isomorphism class of the module P. If (Q,μ) is another representative of $[(P,\varphi)]$ in $g_{n,N}(X)$, there exists especially an isomorphism between P and Q such that β_X is independent of the choice of this representative. Moreover, since for any $M \geq N > n$ in $\mathbb{N}$ and $[(P,\varphi)] \in g_{n,N}(X)$, by definition

$$(f^n_{N,M})_X([(P,\varphi)]) = [(P, \varphi \oplus 0^{M-N})]$$

holds, β_X is also independent of the choice of representative in $\mathrm{Hom}_{\mathsf{PSh}(\mathsf{Aff})}(h_X, \mathrm{Gr}_n)$. Hence β_X is indeed a well-defined map.
The class $\beta := (\beta_X)_{X\in\mathsf{Aff}}$ defines a natural transformation: Consider a morphism of affine schemes $f\colon X = \mathrm{Spec}(S) \to Y = \mathrm{Spec}(R)$ and $[(P,\varphi)] \in g_{n,N}(Y)$. then we obtain the identity

$$\begin{aligned}(f^* \circ \beta_Y)([(N,[(P,\varphi)])]) = f^*([P]) = [P \otimes_R S] &= \beta_X([(N,[(P\otimes_R S, \varphi \otimes_R S)])]) \\ &= (\beta_X \circ h(f)^*)([(N,[(P,\varphi)])]).\end{aligned}$$

The statement of Proposition 3.1.7 remains true if we add a smoothness condition. Hence the natural transformation

$$\beta\colon \quad \mathrm{Hom}_{\mathsf{PSh}(\mathsf{Aff})}(h_{(-)}, \mathrm{Gr}_n) \;\to\; \mathrm{Vect}_n$$

induces by Theorem 3.2.2 and Proposition 3.1.7(ii) a natural transformation

$$\overline{\beta}\colon \quad [-, \mathrm{Gr}_n]_N \;\to\; \mathrm{Vect}_n$$

between the restrictions of the functors to $\mathsf{AffSm}_k^{\mathrm{op}}$, which satisfies $\overline{\beta} \circ \eta = \beta$.

Since any finitely generated projective R-module P of rank n admits for some natural number $N > n$ an epimorphism $\varphi\colon R^N \twoheadrightarrow P$, its isomorphism class has a preimage under $\beta_{\mathrm{Spec}(R)}$. Thus we get from the condition $\overline{\beta} \circ \eta = \beta$, that for any smooth affine k-scheme X, the map $\overline{\beta}_X$ is surjective.

It remains to show injectivity. Let $X = \mathrm{Spec}(R)$ be a smooth affine k-scheme and consider two maps $f, g\colon h_X \to \mathrm{Gr}_n$ with $\beta_X(f) = \beta_X(g)$. Let $r, s > n$ such that f is given by $[(P,\varphi)] \in g_{n,r}(X)$ and g is given by $[(Q,\psi)] \in g_{n,s}(X)$. This means that there exists an isomorphism $Q \xrightarrow{\sim} P$ of R-modules. Thus without loss of generality we can replace Q by P and ψ by $(Q \xrightarrow{\sim} P) \circ \psi$.

Set $N := r + s$ and consider the $R[t]$-linear epimorphisms

$$h_1\colon \quad R[t]^N \to P[t], \qquad e_i \mapsto \begin{cases} \varphi(e_i), & \text{if } 1 \le i \le r, \\ t\psi(e_{i-r}), & \text{if } r+1 \le i \le N, \end{cases}$$
$$h_2\colon \quad R[t]^N \to P[t], \qquad e_i \mapsto \begin{cases} (1-t)\varphi(e_i), & \text{if } 1 \le i \le r, \\ \psi(e_{i-r}), & \text{if } r+1 \le i \le N. \end{cases}$$

Write $f_N\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n, N)$ for the map representing f and $g_N\colon X \to \mathrm{Gr}_{\mathbb{Z}}(n, N)$ for the map representing g. Moreover, write g' for the map $X \to \mathrm{Gr}_n$ and g'_N for the map $X \to \mathrm{Gr}_{\mathbb{Z}}(n, N)$, both determined by $[(P, 0^r \oplus \psi)] \in g_{n,N}(X)$. Since

$$h_1 \otimes_{\mathrm{ev}_0} R = \varphi \oplus 0^s, \qquad h_1 \otimes_{\mathrm{ev}_1} R = h_2 \otimes_{\mathrm{ev}_0} R \qquad \text{and} \qquad h_2 \otimes_{\mathrm{ev}_1} R = 0^r \oplus \psi,$$

it is true by Lemma 3.2.5 that the maps $H_i\colon \mathbb{A}^1_R \to \mathrm{Gr}_{\mathbb{Z}}(n, N)$ given by $[(P[t], h_i)]$ for $i \in \{1, 2\}$ yield a chain of naive $\mathbb{A}^1$-homotopies

$$f_N \overset{H_1}{\Longrightarrow} H_1 \circ \varepsilon_1 \overset{H_2}{\Longrightarrow} g'_N.$$

Thus by Lemma 3.2.4, we conclude that f and g' are naively $\mathbb{A}^1$-homotopic. By Corollary 3.2.6 also g' and g are naively $\mathbb{A}^1$-homotopic, which shows $[f] = [g]$ in $[X, \mathrm{Gr}_n]_N$. □

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