Schemes, functor of points and algebraic vector bundles

Alissa Doggwiler

Universität Regensburg

CHAPTER 1 SCHEMES AND BASIC PROPERTIES

All the rings we will consider in this talk are commutative.

Let (X, O_X) a pair with X a topological space and O_X a sheaf on X.

Definition 1.1: Ringed space

We say that (X, O_X) is a ringed space if O_X is a sheaf of rings.

A morphism between ringed spaces (X, O_X) and (Y, O_Y) is a pair $(f, f^\#)$ where

1. $f : X \to Y$ is a continuous map
2. $f^\# : O_Y \to f_* O_X$ is a morphism of sheaves of rings.

Recall that we define $f_* O_X$ to be a sheaf on Y where $\forall\, U \subseteq Y$ open $f_* O_X(U) = O_X(f^{-1}(U))$. It is called the push-forward of O_X.

Definition 1.2: Locally ringed space

A ringed space (X, O_X) is called a locally ringed space if $\forall x \in X$ the stalk $O_{X,x}$ is a local ring.

A morphism of locally ringed spaces $(f, f^\#) : (X, O_X) \to (Y, O_Y)$ is a morphism of ringed spaces such that for every $x \in X$ the ring morphism $f_x^\# : O_{Y,f(x)} \to O_{X,x}$ is a local morphism of local rings. In other words let $\mathfrak{m}_{X,x}$ and respectively $\mathfrak{m}_{Y,f(x)}$ be the maximal ideals of $O_{X,x}$ and $O_{Y,f(x)}$, then $(f_x^\#)^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,f(x)}$.

The map on stalks $f_x^\# : O_{Y,f(x)} \to O_{X,x}$ comes from the following composition

$$O_{Y,f(x)} = \operatorname*{colim}_{f(x)\in U} O_Y(U) \to \operatorname*{colim}_{f(x)\in U} O_X(f^{-1}(U)) \to \operatorname*{colim}_{x\in V} O_X(V) \cong O_{X,x}$$

Definition 1.3: Affine scheme

A pair (X, O_X) is called an affine scheme if it is a locally ringed space and it is isomorphic, as a locally ringed space, to $(\mathrm{Spec} A, O_{\mathrm{Spec} A})$ for some ring A.

Example 1.4: Construction of a sheaf on Spec(R)

Let R a ring and consider the Zariski topology on Spec(R). Then we can define for every $f \in R$: $O_{\mathrm{Spec}(R)}(D(f)) = R_f$. Taking the restriction maps to just be the obvious ones, it gives rise to a sheaf on Spec(R). Notice that on stalks we have get a local ring: $\forall \mathfrak{p} \in \mathrm{Spec}(R)$ the stalk is of the form: $O_{\mathrm{Spec}(R),\mathfrak{p}} \cong R_{\mathfrak{p}}$

Remark: We define $\mathbb{A}^n_k = \operatorname{Spec}(k[x_1, ..., x_n])$

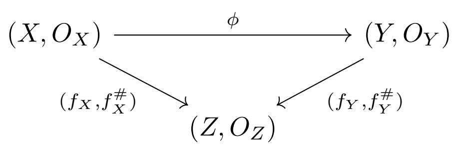
Definition 1.5: Scheme

A scheme (X, O_X) is a locally ringed space, such that there exists an open cover $\bigcup_{i\in I} U_i$ of X such that for every i: $(U_i, O_{U_i} = O_X|_{U_i})_{i\in I}$ is an affine scheme. We write **Sch** for the category of schemes.

Remark: Notice that the topology on X has a basis of affines and hence the restriction of the scheme (X, O_X) to any open subset $(U, O_X|_U)$ remains a scheme.

Definition 1.6: Schemes over a given scheme (Z, O_Z)

A scheme over a given scheme (Z, O_Z), is a scheme (X, O_X) together with a morphism of schemes $(f_X, f_X^\#) : (X, O_X) \to (Z, O_Z)$.
A morphism from (X, O_X) to (Y, O_Y) as schemes over (Z, O_Z) is a morphism of schemes $\phi : (X, O_X) \to (Y, O_Y)$ such that the following diagram commutes



We write $\mathbf{Sch}(Z)$ for the category of schemes over (Z, O_Z).

Remark: When we talk about schemes over a ring R we actually mean schemes over $(\operatorname{Spec}(R), O_{\operatorname{Spec}(R)})$ and we write $\mathbf{Sch}(R)$ for the corresponding category.

Theorem 1.7:

Given R a ring, consider the category of commutative, unital R-algebras $\mathbf{CAlg}_R^{op}$. We have a functor $Spec(-) : \mathbf{CAlg}_R^{op} \to \mathbf{Sch}(R)$ defined by $A \mapsto (Spec(A), O_{Spec(A)})$. It is fully faithful and its essential image corresponds to affine R-schemes.
Furthermore, $Spec(-)$ has a right adjoint given by the global section functor $\Gamma : \mathbf{Sch}(R) \to \mathbf{CAlg}_R^{op}$.

Remark:

1. Recall that we defined in the first talk the category of affine R-schemes to be exactly $\mathbf{CAlg}_R^{\mathrm{op}}$.
2. Fully faithfullness of $\operatorname{Spec}(-)$ gives us for R-algebras A and B:

$$\operatorname{Hom}_{\mathbf{CAlg}_R}(B, A) \cong \operatorname{Hom}_{\mathbf{Sch}(R)}(\operatorname{Spec}(A), \operatorname{Spec}(B))$$

   Hence there is no new data that we can obtain from studing $\operatorname{Hom}_{\mathbf{Sch}(R)}(\operatorname{Spec}(A), \operatorname{Spec}(B))$ instead of $\operatorname{Hom}_{\mathbf{CAlg}_R}(B, A)$.
3. In fact, we can show that we have an adjunction $\operatorname{Spec}(-) : \mathbf{CRing}^{\mathrm{op}} \leftrightarrow \mathbf{Sch} : \Gamma$ which reduces to see that given (X, O_X) a scheme and R a ring, we have $\operatorname{Hom}_{\mathbf{Sch}}((X, O_X), \operatorname{Spec}(R)) \cong \operatorname{Hom}_{\mathbf{CRing}}(R, O_X(X))$. This gives us that $\operatorname{Spec}(\mathbb{Z})$ is a terminal object in **Sch**. Hence the category $\mathbf{Sch}(\mathbb{Z})$ is just **Sch** and $\mathbf{CAlg}_{\mathbb{Z}}^{\mathrm{op}} = \mathbf{CRing}^{\mathrm{op}}$. This gives us a more specific version of the previous theorem for commutative rings.
4. For a more thorough discussion about this theorem, you can take a look at [Aso16] Chapter 7.

Theorem 1.8

There is a fully faithful functor from $\mathbf{Sch}(R)$ in $Fun(\mathbf{CAlg}_R, Set)$

Remark: The proof of this theorem uses the notion of functor of points

Definition 1.9: Functor of points

The functor of points of a scheme X is the functor $h_X : \mathbf{Sch}^{op} \to \text{Set}$ giving for each scheme Y the corresponding set $\text{Mor}(Y, X)$, and for each morphism $f : Y \to Z$ the map of sets $\text{Mor}(Z, X) \to \text{Mor}(Y, X)$ given by precomposition by f.

Since in the theorem we work with schemes over R, we can restrict the study of $\text{Mor}(-, X)$ to the opposite category of affine R-schemes, which as a category is the same as $\mathbf{CAlg}_R$. For more details you can check out Proposition VI-2 in [EH00]

CHAPTER 2

MODULES OVER A SCHEME

Throughout this section let us set (X, O_X) to be a fixed scheme.

Definition 2.1: O_X-modules, locally free sheaves and quasi-coherent sheaves

Let F be a sheaf on X. We say that it is an O_X-module if

1. For every open $U \subseteq X$: $F(U)$ is an $O_X(U)$-module.
2. The restriction maps of F are compatible with the module structure: $\forall\, V \subseteq U$ open in X

$$\begin{array}{ccc} O_X(U) \times F(U) & \longrightarrow & F(U) \\ \downarrow{\scriptstyle \rho^{O_X}_{U,V} \times \rho^F_{U,V}} & & \downarrow{\scriptstyle \rho^F_{U,V}} \\ O_X(V) \times F(V) & \longrightarrow & F(V) \end{array}$$

A morphism $\phi : F \to G$ of O_X-modules, is a morphism of sheaves such that for every open $U \subseteq X$: $\phi(U) : F(U) \to G(U)$ is an $O_X(U)$-module homomorphism.

Furthermore, F is said to be locally free, if there exists an open cover $\bigcup_{i \in I} U_i$ of X such that $F|_{U_i}$ is a free O_{U_i}-module, in other words: $F|_{U_i} \cong \bigoplus_{j \in J} O_{U_i}$.
An invertible sheaf is a locally free sheaf of rank 1.

We say that F is quasi-coherent if there exists an open cover of X by opens U_i such that $F|_{U_i}$ is the cokernel of $\oplus_{i \in I} O_{U_i} \to \oplus_{i \in I} O_{U_i}$.
We write $\text{QCoh}(X)$ for the category of quasi-coherent sheaves on (X, O_X).

Example 2.2: Construction of a module on Spec(R)

Let R a ring and M an R-module. We define $\tilde{M}$ the sheaf associated to M on $\text{Spec}(R)$ following a similar construction than $O_{\text{Spec}(R)}$. Consider $\forall\, f \in R$: $\tilde{M}(D(f)) = M_f$. Taking the restriction maps to just be the obvious ones, it gives rise to a sheaf on $\text{Spec}(R)$ which is in particular a $\text{Spec}(R)$-module.

Note that $\forall \mathfrak{p} \in \mathrm{Spec}(R)$ the stalk is of the form: $\tilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}$

Theorem 2.3

*The functor $\sim$: **RMod** $\to$ Spec$(R)-$Modules sending $M \mapsto \tilde{M}$ is fully faithful and exact.*

Proof:
For fully faithfulness consider M, N two R-modules. We would like to show that

$$\mathrm{Hom}_{\mathbf{RMod}}(M, N) \cong \mathrm{Hom}_{\mathbf{Spec(R)\text{-}M}}(\tilde{M}, \tilde{N})$$

Let $\phi : M \to N$ be an R-module morphism. Then we see that for each $g \in R$ we can construct a well defined R_g-morphism $M_g \to N_g$ where $\frac{m}{g^q} \mapsto \frac{\phi(m)}{g^q}$ and the restriction maps clearly commute. Since the collection of the $D(g)$ form a basis of $\mathrm{Spec}(R)$ this glues to a morphism $\tilde{\phi} : \tilde{M} \to \tilde{N}$.
Taking any $O_{\mathrm{Spec}(R)}$-module morphism $f : \tilde{M} \to \tilde{N}$, we can just evaluate it at $\mathrm{Spec}(R)$ getting an R-linear morphism $f(\mathrm{Spec}(R)) : M \to N$.
We need to show that these two constructions are invertible. Taking global sections, we clearly see that $(\tilde{\phi})(\mathrm{Spec}(R) = D(1)) = \phi$. Now we need to verify that $\sim (f(\mathrm{Spec}(R))) = f$. For this, notice that on $\mathrm{Spec}(R)$ by construction these are the same morphisms. Now for $g \in R$ since f is a sheaf morphism, we have the following commutative diagram on the left

$$\begin{array}{ccc} M & \xrightarrow{f(\mathrm{Spec}(R))} & N \\ \downarrow & & \downarrow \\ M_g & \xrightarrow{f(D(g))} & N_g \end{array} \qquad \begin{array}{ccc} m & \longrightarrow & f(\mathrm{Spec}(R))(m) \\ \downarrow & & \downarrow \\ \frac{m}{1} & \longrightarrow & f(D(g))(m) = \frac{f(\mathrm{Spec}(R))(m)}{1} \end{array}$$

So we see that on $D(g)$ both $\sim (f(\mathrm{Spec}(R)))$ and f agree. Hence they must agree everywhere.

Exactness can be checked on stalks. It follows from the construction of $\tilde{M}$ and commutative algebra.
□

The following lemma now follows quite directly and will be useful for the next theorem:

Lemma 2.4

Let M an R-module. Then $\tilde{M}$ on Spec(R) is quasi-coherent.

Proof:
For any module M we have an exact sequence

$$\bigoplus_{j \in J} R \to \bigoplus_{k \in K} R \to M \to 0$$

Taking $\sim$ over the sequence, noticing that it commutes with direct sums by construction and $\tilde{R} \cong O_{\mathrm{Spec}(R)}$, we see that $\tilde{M}$ is quasi-coherent.
□

In the case where (X, O_X) is an affine scheme, we have a very precise description of the category $\mathrm{QCoh}(X)$:

Theorem 2.5

*The functor **RMod** $\to$ QCoh(SpecR) sending $M \mapsto \tilde{M}$ induces an equivalence of categories with inverse functor: $F \mapsto F(\mathrm{Spec}(R))$.*

Proof:
Fully faithfullness follows from the previous theorem. So there is essential surjectivity left:
We would like to show that $\widetilde{F(X)} \cong F$. For this we will proceed in three steps.

Step 1: Construct a morphism $\widetilde{F(X)} \to F$.
For each $f \in R$ we have a morphism $F(X)_f \to F(D(f))$ coming from the restriction morphism $F(X) \to F(D(f))$ and using that $F(D(f))$ is an R_f-module. This defines a morphism on the basis $D(f)$ and it

clearly commutes with the restiction morphisms, so it glues to a morphism $F(\tilde{X}) \to F$.

Remark: Since F is quasi-coherent on $\mathrm{Spec}(R)$, so there exists an open cover U_i such that $F|_{U_i} \cong \mathrm{coker}(\oplus_{i\in I} O_{U_i} \to \oplus_{i\in I} O_{U_i})$. However, since affines form a basis of $\mathrm{Spec}(R)$, any U_i can be covered by affine opens. By quasi-compactness of $\mathrm{Spec}(R)$ there are only finitely many. We get $\mathrm{Spec}(R) = \bigcup_{i\in I} D(g_i)$ and $F|_{D(g_i)} \cong \mathrm{coker}(\oplus_{i\in I} O_{D(g_i)} \to \oplus_{i\in I} O_{D(g_i)})$. By exactness of $\sim$, $F|_{D(g_i)} \cong \tilde{M}_i$, for M_i some R_{g_i}-module.

Step 2: Injectivity $F(\tilde{X}) \to F$.
We can just verify this on the basis of opens $\{D(f)\}_{f\in R}$. Let $\frac{s}{f^q} \in F(X)_f$ such that $\frac{1}{f^q} s|_{D(f)} = 0$. Since f acts as automorphisms on $F(D(f))$, it implies that $s|_{D(f)} = 0$. We would like to show that there exists $N > 0$ such that $f^N s = 0$ in $F(X)$. This would prove injectivity. Using the remark, we can consider the following diagram

$$\begin{array}{ccc} F(X) & \longrightarrow & F(D(g_i)) \cong M_i \\ \downarrow & & \downarrow \\ F(D(f)) & \longrightarrow & F(D(g_i f)) \cong (M_i)_f \end{array} \qquad \begin{array}{ccc} s & \longrightarrow & s_i := s|_{D(g_i)} \\ \downarrow & & \downarrow \\ s|_{D(f)} = 0 & \longrightarrow & 0 = s_i \end{array}$$

Since $s_i = 0 \in (M_i)_f$ there exists n_i such that $f^{n_i} s_i = 0$ in M_i $\forall i$. Taking $N = \max_i n_i$, which is well defined since our cover is finite, we have $f^N s_i = 0$ $\forall i$. Since $s|_{D(g_i)} = s_i$, the collection $\{s_i\}_{i\in I}$ glues to s. Hence $\{f^N s_i = 0\}_{i\in I}$ glues to $f^N s = 0$. This shows injectivity.

Step 3: Surjectivity $F(\tilde{X}) \to F$.
Let $t \in F(D(f))$. We would like to find a preimage in $F(X)_f$. We can look at the restriction $F(D(f)) \to F(D(g_i f)) \cong (M_i)_f$, where M_i is an R_{g_i}-module, giving $t \mapsto t|_{D(g_i f)} = \frac{s_i}{f^{m_i}}$ for some $s_i \in M_i$. Taking $M = \max_i m_i$, where we again use quasi-compactness of $\mathrm{Spec}(R)$, we get $f^M t|_{D(g_i f)} = f^{M-m_i} s_i$ $\forall i$. Restricting further more, we obtain

$$\begin{array}{ccccc} F(D(f)) & \longrightarrow & F(D(g_i f)) = (M_i)_f & \longrightarrow & F(D(g_i g_j f)) = (M_i)_{fg_j} \\ \downarrow{=} & & & & \downarrow{\cong} \\ F(D(f)) & \longrightarrow & F(D(g_j f)) = (M_j)_f & \longrightarrow & F(D(g_i g_j f)) = (M_j)_{fg_j} \end{array}$$

$$\begin{array}{ccccc} f^M t & \longrightarrow & f^{M-m_i} s_i & \longrightarrow & f^{M-m_i} s_i|_{D(g_i g_j f)} \\ \downarrow{=} & & & & \downarrow{=} \\ f^M t & \longrightarrow & f^{M-m_j} s_j & \longrightarrow & f^{M-m_j} s_j|_{D(g_i g_j f)} \end{array}$$

In $F(D(fg_ig_j))$ we have $f^{M-m_i} s_i|_{D(g_i g_j f)} - f^{M-m_j} s_j|_{D(g_i g_j f)} = 0$. Using the same reasoning as in *Step 2* we conclude that there exists $n(i,j) > 0$ such that $f^{n(i,j)}(f^{M-m_i} s_i|_{D(g_i g_j)} - f^{M-m_j} s_j|_{D(g_i g_j)}) = 0$. Taking $N = \max_{(i,j)} n(i,j)$ we still have $f^N(f^{M-m_i} s_i|_{D(g_i g_j)} - f^{M-m_j} s_j|_{D(g_i g_j)}) = 0$ in $D(g_i g_j)$. By the glueing axioms, the collection $\{f^{N+M-m_i} s_i\}_{i\in I}$ glues to a section s in $F(X)$ such that $s|_{D(g_i)} = f^{N+M} s_i$. In particuar $s|_{D(g_i f)} = f^{N+M} t|_{D(g_i f)}$. So taking $\frac{s}{f^{N+M}}$ we proved surjectivity.

□

CHAPTER 3 PROJECTIVE SPACE AND FUNCTOR OF POINTS

Let us first briefly discuss the construction of the scheme $\mathrm{Proj}(S)$ for S an $\mathbb{N}$-graded ring.

Definition 3.1: $\mathbb{N}$-graded ring

Let S be an $\mathbb{N}$-graded ring. In other words, we fix a decomposition

$$S = \bigoplus_{n \in \mathbb{N}} S_n$$

where $1 \in S_0$, S_n is an abelian group and $\forall n, m \in \mathbb{N}$ $S_n S_m \subseteq S_{n+m}$.
We say that an element $f \in S$ is homogeneous if $f \in S_n$ for some n.
An ideal $I \subseteq S$ is called homogeneous if we can decompose it as $I = \bigoplus_{n \in \mathbb{N}} I \cap S_n$.
We denote by $S_+ = \bigoplus_{n>0} S_n$ the irrelevant ideal.

Definition 3.2: $\mathrm{Proj}(S)$ as a topological space

We set $\mathrm{Proj}(S) = \{\mathfrak{p} \subseteq S | \mathfrak{p} \text{ homogeneous prime ideal and } S_+ \not\subseteq \mathfrak{p}\}$.
Let I a homogeneous ideal, we define $V_+(I) = \{\mathfrak{p} \in \mathrm{Proj}(S) | I \subseteq \mathfrak{p}\}$. We can verify that

1. $V_+(S_+) = \emptyset$
2. $V_+(0) = \mathrm{Proj}(S)$
3. For any two homogeneous ideals I, J of S: $V_+(I) \cup V_+(J) = V_+(IJ)$.
4. For any family $\{I_k\}_{k \in K}$ of homogeneous ideals $\bigcap_{k \in K} V_+(I_k) = V_+(\sum_{k \in K} I_k)$

These sets define a topology on $\mathrm{Proj}(S)$ where they are closed.

Lemma 3.3

Let $f \in S_+$ a homogeneous element, we define $D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(S) | f \notin \mathfrak{p}\}$. We see that

1. *$D_+(f)$ is open in $\mathrm{Proj}(S)$.*
2. *For $g \in S$ homogeneous: $D_+(f) \cap D_+(g) = D_+(fg)$*
3. *The collection of sets $D_+(f)$ forms a basis of the topology of $\mathrm{Proj}(S)$.*

Definition 3.4: Construction of a sheaf on $\mathrm{Proj}(S)$

$\forall f \in S_+$ homogeneous, we define $S_{(f)} = \left(S_f\right)_0 = \{\frac{a}{f^q} | \, a \in S \text{ homogeneous } \& \ \deg(a) = q \cdot \deg(f)\}$.
We consider the collection of affine schemes $\{D_+(f) \cong \mathrm{Spec}(S_{(f)})\}_{f \in S \text{ homog.}}$ and glue them along

morphisms that we will not explicit here to obtain a scheme on $\text{Proj}(S)$.

More specificly we will study the following scheme:

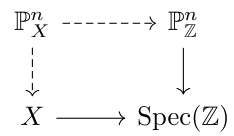
Example 3.5: $\mathbb{P}^n_{\mathbb{Z}}$

We define $\mathbb{P}^n_{\mathbb{Z}} := \text{Proj}(\mathbb{Z}[x_0, ..., x_n])$, where the variables $x_0, ..., x_n$ have the obvious degree 1. In this case, notice that the collection $\{D_+(x_i)\}_{i=0}^n$ covers $\mathbb{P}^n_{\mathbb{Z}}$ since a homogeneous prime which is in $\left(\bigcup_{i=0}^n D_+(x_i)\right)^c$ must contain $(x_0, ..., x_n) = \mathbb{Z}[x_0, ..., x_n]_+$ which is impossible by definition.

$\mathbb{P}^n_{\mathbb{Z}}$ can be seen as a glueing of the collection $\{D_+(x_i)\}_{i=0}^n$ along their intersections $\{D_+(x_i x_j)\}_{i,j=0}^n$. By doing a quick computation we easily notice that these affines $D_+(x_i)$ are just $\mathbb{A}^n_{\mathbb{Z}}$.

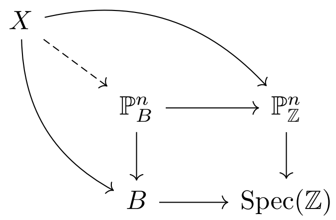
Definition 3.6: $\mathbb{P}^n_X$

In general, for a scheme (X, O_X) we define $\mathbb{P}^n_X$ to be the following pullback in the category of schemes



Recall that since $\text{Spec}(\mathbb{Z})$ is terminal, we can always find maps to $\text{Spec}(\mathbb{Z})$.

We would like to describe in this section, what $\text{Mor}(X, \mathbb{P}^n_B)$ is for given schemes X and B. Using the universal property of the pullback, we see that for a given morphism $X \to B$, it is sufficient to understand $\text{Mor}_B(X, \mathbb{P}^n_{\mathbb{Z}})$.



Theorem 3.7

$Mor(X, \mathbb{P}^n_{\mathbb{Z}}) \cong \{Epimorphisms\ O_X^{n+1} \to P \text{ with } P \text{ an invertible } O_X\text{-mod.}\}/\{O_X(X)^\times \text{ acting on } P\}$

Remark: We follow [EH00] for the proof of this theorem.
Proof:
We can directly reduce to the case where X is affine, since morphisms from X can just be defined locally on affines and then glued together. So let us prove following proposition:

Proposition 3.8

Let T a ring, then we have the following bijection:

$$Mor(Spec(T), \mathbb{P}^n_{\mathbb{Z}}) \cong \{Surjections\ T^{n+1} \to P \text{ where } P \text{ is an invertible } T\text{-module}\}/\{isomorphism\}$$

Proof:
Suppose we have a morphism $\phi : \text{Spec}(T) \to \mathbb{P}^n_{\mathbb{Z}}$. We have a covering of $\mathbb{P}^n_{\mathbb{Z}}$ given by $D_+(x_i)$ for $i = 0, ..., n+1$. Taking preimages, we have a covering of $\text{Spec}(T) = \bigcup_{i=0}^n U_i$ and morphisms $U_i \to \text{Spec}(\mathbb{Z}[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}])$. Using the adjunction in the Remark of Theorem 1.7 we get a ring map $\mathbb{Z}[\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}] \to O_{\text{Spec}(T)}(U_i)$. This corresponds precisely to a choice of elements $(t_{i0}, ..., t_{ii} = 1, ..., t_{in})$ in $O_{\text{Spec}(T)}(U_i)$ Notice that on the intersection $U_i \cap U_j$ we have that t_{ij} is a unit in $O_{\text{Spec}(T)}(U_i \cap U_j)$ and $t_{il} = t_{ij} t_{jl}$ in $O_{\text{Spec}(T)}|_{U_i \cap U_j}$ using the induced ring maps on intersections. We now need to make us of the following result

Theorem 3.9

Let (X, O_X) a scheme and F an O_X-module. Then $Hom_{O_X}(O_X, F) \cong F(X)$.

Remark: The idea is that O_X-homomorphisms are entirely determined by the choice of the image of 1 in $O_X(X)$, this is similar to the case where we consider R-module morphisms from R into any R-module.

Corollary 3.10

Let (X, O_X) a scheme and F an O_X-module. Then $Hom_{O_X}(O_X^n, F) \cong \{(s_1, ..., s_n) \in F(X)\}$.

We see that this choice of sections give us an $O_{\mathrm{Spec}(T)}|_{U_i}$-morphism $O_{\mathrm{Spec}(T)}|_{U_i}^{n+1} \to O_{\mathrm{Spec}(T)}|_{U_i}$. By the properties mentionned above about the t_{ij} we see that these maps agree on intersections and glue to a morphism $O_{\mathrm{Spec}(T)}^{n+1} \to F$, where F is an invertible $\mathrm{Spec}(T)$-module. To show that on global sections it is an epimorphism, we can reduce to stalks and use that $t_{ii} = 1$ as well as the fact that any $\mathfrak{p}$ must be in some U_i:

$$\begin{array}{ccc} O_{\mathrm{Spec}(T)}(U_i)^{n+1} & \longrightarrow & O_{\mathrm{Spec}(T)}(U_i) \\ \downarrow & & \downarrow \\ T_{\mathfrak{p}}^{n+1} & \longrightarrow & T_{\mathfrak{p}} \end{array} \qquad \begin{array}{ccc} 1_i & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \alpha & \longrightarrow & 1 \end{array}$$

The lower horizontal map must be a surjection since it is a $T_{\mathfrak{p}}$-module morphism. This shows that $O_{\mathrm{Spec}(T)}^{n+1} \to F$ is an epimorphism. Using exactness of the global section functor on $\mathrm{QCoh}(\mathrm{Spec}(T))$ we conclude that we have a surjection $T^{n+1} \to P$, where P is an invertible module.

Now let a sujective T-module morphism $T^{n+1} \to P$ where P is locally of rank 1. Let us denote $p_0, ..., p_n$ the images of $1_0, ..., 1_n \in T^{n+1}$. Consider I_j the annihilator of P/Tp_j. We can prove that the collection of $U_j = V(I_j)^c$ forms an open cover of $\mathrm{Spec}(T)$. Let $\mathfrak{p} \in \mathrm{Spec}(T)$, since we know that P is invertible, $T_{\mathfrak{p}} \cong P_{\mathfrak{p}}$. There must be $p_j \notin \mathfrak{p}P_{\mathfrak{p}}$. Indeed, otherwise $\mathfrak{p}P_{\mathfrak{p}} = P_{\mathfrak{p}}$ but then $0 = P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}} \cong T_{\mathfrak{p}}/\mathfrak{p}T_{\mathfrak{p}} \cong T/\mathfrak{p}$ which cannot be the case. Furthermore, we can suppose that $P_{\mathfrak{p}}$ is generated by p_j. Indeed, let f be the generator of $P_{\mathfrak{p}}$. Then $\frac{t}{q}f = p_j$. Since $p_j \notin \mathfrak{p}P_{\mathfrak{p}}$, $t \notin \mathfrak{p}$ so it is invertible and hence p_j generates $P_{\mathfrak{p}}$. Since p_j generates $P_{\mathfrak{p}}$, there must be some $a_i \in T \setminus \mathfrak{p}$ such that $a_i p_i$ is a multiple of p_j. Taking $a := \prod a_i \in T \setminus \mathfrak{p}$ we get that $aP \subseteq Tp_j$. Hence $a \in I_j$ and so $\mathfrak{p} \in U_j$.
Consider now the map $T \to P$ such that $1 \mapsto p_j$. Sheafifying, we can see this map as a map of $O_{\mathrm{Spec}(T)}$-modules. We can check that the restriction to U_j is an isomorphism by verifying this on stalks. Coming back to our sheaf map $\tilde{T}^{n+1} \to \tilde{P}$, we see that on U_j we have a morphism $\tilde{T}^{n+1}|_{U_j} \to \tilde{P}|_{U_j} \cong \tilde{T}|_{U_j}$ which on U_j gives a matrix $(t_{j0}, ..., t_{jj} = 1, ..., t_{jn})$. Going through the same steps as beofre, we get maps $\mathrm{Spec}(T)|_{U_j} \to \mathbb{A}_{\mathbb{Z}}^n$ which agree on intersections, hence they glue to $\mathrm{Spec}(T) \to \mathbb{P}_{\mathbb{Z}}^n$. □

CHAPTER 4
VECTOR BUNDLES

Definition 4.1: Vector bundle

Let X,Y be two schemes. We say that Y is a rank n vector bundle over X if there exists a morphism $f : Y \to X$ and an open cover $\{U_i\}_{i\in I}$ of X, such that

1. For each $i \in I$ we have an isomorphism of schemes $\phi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i} = \mathrm{Spec}(\mathbb{Z}[x_1, ..., x_n]) \times_{\mathbb{Z}} U_i$
2. For every open affine $\mathrm{Spec}(A) = V \subseteq U_i \cap U_j$ the restricted composition $\phi_j \circ \phi_i^{-1} : \mathbb{A}^n_V \to \mathbb{A}^n_V = \mathrm{Spec}(A[x_1, ..., x_n])$ is a linear map on the global sections, in the sense that we have a linear automorphism ϕ on $A[x_1, ..., x_n]$ such that $\phi(a) = a \ \forall a \in A$ and $\phi(x_i) = \sum_{j=1}^n a_{ij}x_j$ for some $a_{ij} \in A$.

We write $(Y, f, \{U_i\}, \{\phi_i\})$ for this vector bundle.
An isomorphism of vector bundles $(Y, f, \{U_i\}, \{\phi_i\})$ and $(Y', f', \{U'_i\}, \{\phi'_i\})$ over X is an isomorphism $g : Y \to Y'$ of schemes such that $f' \circ g = f$ and $(Y, f, \{U_i\} \cup \{U'_i\}, \{\phi_i\} \cup \{\phi'_i \circ g\})$ is a vector bundle over X.

For a scheme X, we write $V_n(X)$ for the isomorphism classes of vector bundles of rank n over X.

Remark: When we write $\mathbb{A}^n_{U_i}$ we actually mean the pullback $\mathrm{Spec}(\mathbb{Z}[x_1, ..., x_n]) \times_{\mathbb{Z}} U_i$ like in projective case, see Definition 3.6:

$$\begin{array}{ccc} \mathbb{A}^n_{U_i} & \dashrightarrow & \mathbb{A}^n_{\mathbb{Z}} \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

Lemma 4.2

There is a one to one correspondence between isomorphism classes of vector bundles of rank n over X and isomorphism classes of locally free sheaves of rank n over X.

Proof: Hartshorne exercise II.5.18. [Har77]

We will now use this result to study vector bundles over $\mathbb{P}^1_k$, following the steps in [Aso16]

Step 1:

Lemma 4.3

We have a bijection $V_n(\mathbb{P}^1_k) \cong GL_n(k[t]) \setminus GL_n(k[t, t^{-1}])/GL_n(k[t^{-1}])$

Remark:

1. The right side are classes of matrices in $GL_n(k[t,t^{-1}])$ up to left multiplication by $GL_n(k[t])$ and right multiplication by $GL_n(k[t^{-1}])$.
2. Since affines from a basis of $\mathbb{P}^1_k$, a locally free sheaf of rank n over $\mathbb{P}^1_k$, restricts to a locally free sheaf on each of the two copies of $\mathbb{A}^1_k$ in $\mathbb{P}^1_k$. Using the following lemma we see that any locally free sheaf of rank n over $\mathbb{P}^1_k$ is just the glueing of two free sheaves of rank n on $\mathbb{A}^1_k$ along their intersection in $\mathbb{P}^1_k$.

Lemma 4.4

Any locally free sheaf of rank n over $\mathbb{A}^n_k$ is free.

Proof of Lemma 4.3:
Let us start by constructing a locally free sheaf of rank n over $\mathbb{P}^1_k$ given a matrix in $GL_n(k[t,t^{-1}])$. The two affines covering $\mathbb{P}^1_k$ are $D_+(x_0)$ and $D_+(x_1)$. As discussed before, we know that $D_+(x_0) \cong \mathrm{Spec}(k[x_0,x_1]_{(x_0)}) \cong k\left[\frac{x_1}{x_0}\right] \cong k[t]$, $D_+(x_1) \cong \mathrm{Spec}(k[t^{-1}])$ and $D_+(x_0) \cap D_+(x_1) = D_+(x_0x_1) \cong \mathrm{Spec}(k[t,t^{-1}])$. Now taking two rank n free modules P_+ and P_- on respectively $k[t]$ and $k[t^{-1}]$ they give rise to two free sheaves of rank n on $\mathrm{Spec}(k[t])$ and $\mathrm{Spec}(k[t^{-1}])$. Since we want to glue them on $\mathrm{Spec}(k[t,t^{-1}])$, we extend them to modules on $k[t,t^{-1}]$:

$$P_+ \otimes_{k[t]} k[t,t^{-1}] \text{ and } P_- \otimes_{k[t^{-1}]} k[t,t^{-1}]$$

These remain free of rank n. We can choose a basis for each module: $\{e_i^+\}_{i=1}^n$ and $\{e_i^-\}_{i=1}^n$ and use a matrix in $GL_n(k[t,t^{-1}])$ to define an isomorphism

$$P_+ \otimes_{k[t]} k[t,t^{-1}] \to P_- \otimes_{k[t^{-1}]} k[t,t^{-1}]$$

Just pay attention that we want to send $t \mapsto t^{-1}$ to respect the module structures of both sides.
Different choices of bases on both sides give different isomorphisms. However, the two different glueings we obtain are isomorphic. This comes from the fact that if there is an open cover on which there are isomorphisms between two sheaves and such that these morphisms agree on intersection, then the sheaves are isomorphic. This is how we see that multiplying our matrix in $GL_n(k[t,t^{-1}])$ on the left by a matrix of $GL_n(k[t])$ or on the right by a matrix of $GL_n(k[t^{-1}])$ does not change the isomorphism class of our locally free sheaf. So we get an injective map

$$GL_n(k[t]) \setminus GL_n(k[t,t^{-1}])/GL_n(k[t^{-1}]) \to V_n(\mathbb{P}^1_k)$$

Now for a given locally free sheaf of rank n on $\mathbb{P}^1_k$, we know that on each open affine $D_+(x_0)$ and $D_+(x_1)$, our sheaf corresponds to a free module P_+ respectively P_- of rank n. Since on $D_+(x_0x_1)$ they must agree, choosing bases, we get a matrix in $GL_n(k[t,t^{-1}])$.

□

Definition 4.5: Clutching function

The matrix class associtated to a bundle is called the clutching function.

Our bijection $GL_n(k[t]) \setminus GL_n(k[t,t^{-1}])/GL_n(k[t^{-1}]) \cong V_n(\mathbb{P}^1_k)$ gives us a first description of $V_n(\mathbb{P}^1_k)$. It is however not really intuitive what exactly these matrix classes are. To get a better idea, we can consider the following lemma

Step 2:

Lemma 4.6

Any matrix $A \in GL_n(k[t,t^{-1}])$ can be written as a product VDU where $V \in GL_n(k[t^{-1}]), U \in$

$GL_n(k[t])$ and D is a diagonal matrix

$$D = \begin{bmatrix} t^{a_1} & & & \\ & t^{a_2} & & \\ & & \dots & \\ & & & t^{a_n} \end{bmatrix}$$

with integers $a_1 \geqslant a_2 \geqslant ... \geqslant a_n$

Proof: We will not prove this here, but you can find the main steps in [Aso16] Proposition 8.1.1.4.

This result helps us give the following description of locally free sheaves of rank n over $\mathbb{P}^1_k$:

Step 3:

Theorem 4.7

Any vector bundle of rank n on $\mathbb{P}^1_k$ is isomorphic to a vector bundle of the form $O(a_1) \oplus O(a_2) ... \oplus O(a_n)$ for a unique sequence of integers $a_1 \geqslant a_2 \geqslant ... \geqslant a_n$.

Definition 4.8: Line bundle $O(n)$

There are several definitions we can give of $O(n)$. We choose to present this one there: it is a line bundle on $\mathbb{P}^1_k$, i.e. a vector bundle of rank 1, such that the clutching function is exactly given by t^n.

Proof: The isomorphism is direct. We should prove unicity of the integers $a_1 \geqslant a_2 \geqslant ... \geqslant a_n$, but this follows by some computations we will not expand here. Feel free to check them out in [ADav] Theorem 7.3.

□

This is everything we wanted to say about $V_n(\mathbb{P}^1_k)$. To conclude this talk, we would like to briefly discuss $\mathbb{A}^1_k$-invariance:

Example 4.9: Pullback of vector bundles

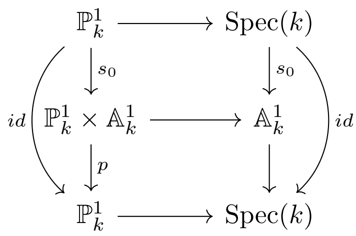
The map $V_2(\mathbb{P}^1_k) \to V_2(\mathbb{P}^1_k \times \mathbb{A}^1_k)$ given by the pullback of vector bundles is not surjective.

This map uses the following pullback constructions:

$$\begin{array}{ccc} \mathbb{P}^1_k \times \mathbb{A}^1_k & \longrightarrow & \mathbb{A}^1_k \\ \downarrow p & & \downarrow \\ \mathbb{P}^1_k & \longrightarrow & \mathrm{Spec}(k) \end{array} \qquad \begin{array}{ccc} p^*(V) & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathbb{P}^1_k \times \mathbb{A}^1_k & \xrightarrow{p} & \mathbb{P}^1_k \end{array}$$

Notice that $\mathbb{P}^1_k \times \mathbb{A}^1_k = \mathrm{Proj}(k[x, t_0, t_1])$ where x has degree 0 and t_1, t_2 both have degree 1. Hence a cover of $\mathbb{P}^1_k \times \mathbb{A}^1_k$ is just given by $D_+(t_0) \cong \mathrm{Spec}(k[x, t]), D_+(t_1) \cong \mathrm{Spec}(k[x, t^{-1}])$ with intersection $D_+(t_0, t_1) \cong \mathrm{Spec}(k[x, t, t^{-1}])$. Repeating the same steps as in the theorem, we can see that giving a matrix in $GL_2(k[x, t, t^{-1}])$ gives a vector bundle of rank 2 on $\mathbb{P}^1_k \times \mathbb{A}^1_k$. Considering for example $\begin{bmatrix} t & 0 \\ x & t^{-1} \end{bmatrix}$ we can prove that the associated vector bundle cannot be a pullback of a vector bundle over $\mathbb{P}^1_k$. Indeed, its fiber should be the same over any x, which here is not the case. See explainations below. Set $x = 0$ we get the bundle $O(1) \oplus O(-1)$. Take $x = 1$ we get a matrix equivalent to the identity, up to multiplication on the left by a matrix of $GL_n(k[t])$ or on the right by a matrix of $GL_n(k[t^{-1}])$, giving the bundle $O(0) \oplus = O(0)$.

We can show that any bundle which is a pullback, needs to have the same fibers. Let us show this for the fibers over 0 and 1. Consider $s_0 : \mathrm{Spec}(k) \to \mathbb{A}^1$ coming from $ev_0 : k[x] \to k$ and s_1 coming from the evaluation at 1. We can construct maps taking pullbacks



On the right vertical composition we obtain the identity, by just checking on rings that the composition of the inclusion with the evaluation at 0 is really the identity on k: $k \to k[x] \to k$. The map $s_0 : \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{A}^1_k$ is also the one induced by setting $x = 0$ on rings. We can proceed the same way for s_1. Then if $p^*(V)$ is a vector bundle on $\mathbb{P}^1_k \times \mathbb{A}^1_k$ pulledback from a vector bundle V on $\mathbb{P}^1_k$ we see that

$$s_0^*(p^*(V)) = (ps_0)^*(V) = V = (ps_1)^*(V) = s_1^*(p^*(V))$$

But taking s_0^* is preciely setting x to 0 in the clutching function. The sames goes for s_1. This shows that the two cluching functions should give the same vector bundle if it was pulledback from a vector bundle on $\mathbb{P}^1_k$. Hence the map $V_2(\mathbb{P}^1_k) \to V_2(\mathbb{P}^1_k \times \mathbb{A}^1_k)$ is not surjective.

CHAPTER 5 BIBLIOGRAPHY

[ADav] Ari Davidovsky, *The Classification of Vector Bundles over P1*, http://math.uchicago.edu/ may/REU2021/REUPapers/Davidovsky.pdf

[Aso16] A. Asok, *Algebraic geometry from an A1-homotopic viewpoint*, 2016, https://dornsife.usc.edu/aravind-asok/wp-content/uploads/sites/213/2023/10/MATH599.S16.pdf

[EH00] D. Eisenbud and J. Harris, The Geometry of Schemes, 2000, https://www.maths.ed.ac.uk/ v1ranick/papers/eisenbudharris.pdf

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer, 1977