SS 2025 SEMINAR — THE BLOCH–KATO CONJECTURE

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Last semester we presented a proof of the Bloch–Kato conjecture assuming:

(1) The computation of the slices of algebraic cobordism.
(2) The existence of special summands of the motives of v_n-varieties called *Rost motives*, and a closely related exact sequence used in the construction of norm varieties.

The goal of this seminar is to establish these results and thereby complete the proof of the Bloch–Kato conjecture. The main ingredient for both results is the determination of the *motivic Steenrod algebra*, which is the algebra of bistable operations in motivic cohomology with coefficients in $\mathbb{Z}/l$.

Remark. Most results in this seminar hold over fields of arbitrary characteristic $\neq l$, but only the case of characteristic zero is needed for the Bloch–Kato conjecture. Therefore, we will freely use both resolution of singularities and Betti realization when convenient.

1. Operations in motivic cohomology

We introduce Steenrod operations in motivic cohomology and describe the resulting *motivic Steenrod algebra* $\mathcal{A}^{**}$ following [Voe03b].

1. **Power operations in motivic cohomology (April 29)** [Voe03b, §5–6]
Review Voevodsky's definition of motivic cohomology and of motivic Eilenberg–Mac Lane spaces. Explain the construction of the refined lth power map [Voe03b, §5]. Compute the motives and motivic cohomology with $\mathbb{Z}/l$-coefficients of the classifying stacks $B\mu_l$ and $B\Sigma_l$ [Voe03b, §6][Spi18, §10.2], and hence define the Steenrod operations P^i and B^i.
2. **The motivic Steenrod algebra (May 6)** [Voe03b, §7–11]
Go over the basic properties of P^i and B^i [Voe03b, 9.2, 9.4, 9.5, 9.6, 9.7, 9.8, 9.9]; in particular that they are $\mathbb{P}^1$-stable operations, that $B^i = \beta P^i$, and the Cartan formula. State the Adem relations [Voe03b, Theorem 10.3] (some typos are corrected in [HKØ17, §5.1]) and describe the structure of the motivic Steenrod algebra $\mathcal{A}^{**}$ as a cocommutative Hopf algebra [Voe03b, §11].
3. **The Milnor operations (May 13)** [Voe03b, §12–14]
Define the elements ξ_i and τ_i and describe the structure of the dual motivic Steenrod algebra $\mathcal{A}_{**}$ [Voe03b, §12] [Hoy15, §5.3]. Introduce the motivic cohomology operations $\rho(E,R)$ and in particular the Milnor operations Q_i and their properties [Voe03b, §13]. Compute the effect of these operations on characteristic classes [Voe03b, Theorem 14.2].

2. Motivic Eilenberg–Mac Lane spaces

We compute the motives of motivic Eilenberg–Mac Lane spaces following [Voe10b]. Among other things, this leads to an identification of the motivic Steenrod algebra $\mathcal{A}^{**}$ from the previous section with the bigraded endomorphisms of the motivic spectrum $\mathrm{H}\mathbb{Z}/l$ and of its dual $\mathcal{A}_{**}$ with the bigraded homotopy groups of $\mathrm{H}\mathbb{Z}/l \otimes \mathrm{H}\mathbb{Z}/l$.

Remark. In the paper [Voe10b], Voevodsky heavily uses the formalism of "simplicial radditive functor" developed in [Voe10c]. We suggest ignoring this reference, as it is merely an implementation of the now standard formalism of *nonabelian derived categories/animation.*

4. **Symmetric powers of Tate motives (May 20)** [Voe10b, §2]
The goal of this talk is to explain [Voe10b, Theorem 2.76]. In particular, the symmetric power functors S^n_{tr} on the unstable category of motives $\mathrm{H}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Cor}(\mathcal{C}, R))$ and the subcategory of split proper Tate motives should be explained. The main ingredient is the computation of the symmetric powers of Tate motives in [Voe10b, Theorem 2.58].
5. **Motives of Eilenberg–Mac Lane spaces (May 27)** [Voe10b, §3]
Review the motivic Dold–Thom theorem describing the motivic Eilenberg–Mac Lane space $K(\mathbb{Z}[1/c], p, q)$ in terms of symmetric powers of schemes [Voe10b, Theorem 3.7]. Then explain how to use it to compute the $\mathbb{Z}[1/c]$-motive of $K(A, p, q)$ [Voe10b, Theorem 3.25]. Deduce [Voe10b, Corollary 3.33] using the main result of the previous talk. Finally, explain why the motivic Steenrod operations generate all $\mathbb{P}^1$-stable motivic cohomology operations by reduction to $k = \mathbb{C}$ and comparison with topology [Voe10b, Theorem 3.49].

3. The slices of algebraic cobordism and Morava K-theory

Using our understanding of $\mathrm{H}\mathbb{Z}/l \otimes \mathrm{H}\mathbb{Z}/l$, we prove the Hopkins–Morel isomorphism

$$\mathrm{H}\mathbb{Z} = s_0(\mathrm{MGL}) \simeq \mathrm{MGL}/(x_1, x_2, \dots)$$

[Hoy15], which allows us to describe the whole slice filtration of MGL [Spi10].

6. **The motivic cohomology of quotients of MGL (June 3)** [Hoy15, §6]
Explain the computation of the motive/motivic cohomology of quotients of MGL by suitable sequences of elements of the Lazard ring (such as the algebraic Morava K-theory spectrum). In particular, for a suitable family of generators $(x_i)_{i\geq 1}$ of the Lazard ring, the $\mathbb{Z}/l$-motivic cohomology of $\mathrm{MGL}/(l, x_1, x_2, \dots)$ is the motivic Steenrod algebra $\mathcal{A}^{**}$. A key lemma is the relation between the Milnor operation Q_n and the cobordism element v_n [Hoy15, Lemma 6.13].
7. **The Hopkins–Morel isomorphism (June 17)** [Hoy15, §3, §7] [Spi10]
Using the cohomology computation of the previous talk and some facts about Morel's homotopy t-structure, deduce the Hopkins–Morel isomorphism $\mathrm{MGL}/(x_1, x_2, \dots) = \mathrm{H}\mathbb{Z}$ (over fields of characteristic zero) [Hoy15, Theorem 7.12]. Then explain how to compute the slice filtration of MGL in terms of quotients by the x_i's, following [Spi10]. Observe in particular that Q_n is the first nontrivial differential in the slice spectral sequence for algebraic Morava K-theory $\mathrm{K}(n)$.

4. Rost motives

We complete the proof of the Bloch–Kato conjecture by constructing Rost motives, following Voevodsky's paper [Voe11].

Remark. Voevodsky uses throughout the formalism of motives over simplicial schemes developed in [Voe10a]. We suggest ignoring this reference as it is simply an outdated way of constructing the right Kan extension of $\mathrm{DM}^{\mathrm{eff}}$ from schemes to presheaves. An "embedded simplicial scheme" can be understood as a subpresheaf of $\mathrm{Spec}(k)$.

8. **The motivic degree theorem (June 24)** [Voe11, §4]
The goal of this talk is to prove [Voe11, Theorem 4.4]. Recall first the construction of the motivic Pontryagin–Thom collapse from [Voe03a, §2]. A key ingredient is a vanishing

result for Margolis homology, which we already established last semester using Morava K-theory.

9. **A uniqueness theorem for motivic Steenrod operations (July 1)** [Voe11, §2]
Prove [Voe11, Theorem 2.1], which characterizes
$$\beta P^n\colon \mathrm{H}^{2n+1,n}(-,\mathbb{Z}/l) \to \mathrm{H}^{2nl+2,nl}(-,\mathbb{Z}/l)$$
as the essentially unique cohomology operation that vanishes on suspensions.
10. **Cohomology operations from symmetric powers (July 8)** [Voe11, §3]
Prove [Voe11, Lemma 3.1] to construct the cohomology operations
$$\phi_i\colon \mathrm{H}^{2q+1,p}(-,\mathbb{Z}/l) \to \mathrm{H}^{2(i+1)q+2,(i+1)p}(-,\mathbb{Z}/l)$$
for $i < l$. Using the result of the previous talk, show that $\phi_{l-1} = \beta P^n$ when $p = q = n$.
11. **Generalized Rost motives (July 15)** [Voe11, §5]
For any sieve $\mathcal{X} \subset \mathrm{Spec}(k)$ containing a v_n-variety and any nonzero cohomology class $\delta \in \mathrm{H}^{n+1,n}(\mathcal{X},\mathbb{Z}/l)$, construct the associated *Rost motive* $M(\mathcal{X},\delta)$. Show that $M(\mathcal{X},\delta)$ is a direct summand of $M(X)$ for any v_n-variety X in $\mathcal{X}$.
12. **The Bloch–Kato conjecture (July 22)** [Voe11, §6]
Explain the proof of [SJ06, Theorem A.1], which is the final ingredient in the proof of the Bloch–Kato conjecture in [Voe11, §6]. This talk should also give a high-level summary of the whole proof, recalling how all the pieces fit together.

References

[HKØ17] M. Hoyois, S. Kelly, and P. A. Østvær, *The motivic Steenrod algebra in positive characteristic*, J. Eur. Math. Soc. **19** (2017), no. 12, pp. 3813–3849, preprint arXiv:1305.5690

[Hoy15] M. Hoyois, *From algebraic cobordism to motivic cohomology*, J. reine angew. Math. **702** (2015), pp. 173–226, preprint arXiv:1210.7182

[SJ06] A. Suslin and S. Joukhovitski, *Norm varieties*, J. Pure Appl. Algebra **206** (2006), no. 1-2, pp. 245–276

[Spi10] M. Spitzweck, *Relations between slices and quotients of the algebraic cobordism spectrum*, Homology, Homotopy Appl. **12** (2010), no. 2, pp. 335–351, `http://projecteuclid.org/getRecord?id=euclid.hha/1296223886`

[Spi18] M. Spitzweck, *A commutative $\mathbf{P}^1$-spectrum representing motivic cohomology over Dedekind domains*, Mém. Soc. Math. Fr. **157** (2018), preprint arXiv:1207.4078

[Voe03a] V. Voevodsky, *Motivic cohomology with $\mathbf{Z}/2$-coefficients*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, pp. 59–104

[Voe03b] ———, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. (2003), no. 98, pp. 1–57, `http://dx.doi.org/10.1007/s10240-003-0009-z`

[Voe10a] ———, *Motives over simplicial schemes*, J. K-Theory **5** (2010), pp. 1–38

[Voe10b] ———, *Motivic Eilenberg-Maclane spaces*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 112, pp. 1–99, `http://dx.doi.org/10.1007/s10240-010-0024-9`

[Voe10c] ———, *Simplicial radditive functors*, J. K-Theory **5** (2010), no. 2, pp. 201–244, `http://dx.doi.org/10.1017/is010003026jkt097`

[Voe11] ———, *On motivic cohomology with $\mathbf{Z}/l$-coefficients*, Ann. of Math. (2) **174** (2011), no. 1, pp. 401–438, `http://dx.doi.org/10.4007/annals.2011.174.1.11`