SoSe 25 Algebraic K-theory. Exercise sheet 10

Exercise 1. Let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subset \mathcal{A}$ a full subcategory containing 0 and closed under extensions. Show that $\mathcal{C}$ admits an exact structure in which a morphism is an admissible monomorphism (resp. an admissible epimorphism) if and only if it is a monomorphism (resp. an epimorphism) in $\mathcal{A}$ whose cokernel (resp. kernel) belongs to $\mathcal{C}$. Moreover, the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{A}$ is then exact.

Exercise 2. Let $\mathrm{Ab}^{\mathrm{fg}}$ be the exact category of finitely generated abelian groups and $\mathrm{Ab}^{\mathrm{fg}}_{\mathrm{tor}} \subset \mathrm{Ab}^{\mathrm{fg}}$ the exact subcategory of torsion abelian groups.

(a) Show that $K_0(\mathrm{Ab}^{\mathrm{fg}}) \simeq \mathbb{Z}$.

(b) Determine $K_0(\mathrm{Ab}^{\mathrm{fg}}_{\mathrm{tor}})$.

Exercise 3. Let $\mathcal{C}$ be an exact category. Construct an equivalence between $Q(\mathcal{C})$ and $Q(\mathcal{C}^{\mathrm{op}})$.

Exercise 4. Let $p\colon \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration whose fibers are groupoids. Suppose that, for every morphism $u\colon b \to b'$ in $\mathcal{B}$, the pullback functor $u^*\colon p^{-1}(b') \to p^{-1}(b)$ is an equivalence. Show that $N(p)\colon N(\mathcal{E}) \to N(\mathcal{B})$ is a Kan fibration.

Remark. This generalizes the fact that the nerve of a groupoid is a Kan complex (take $\mathcal{B} = *$).