SoSe 25 Algebraic K-theory. Exercise sheet 11

Exercise 1. Apply Quillen's Theorem A to the inclusion of categories with one object $B\mathbb{N} \to B\mathbb{Z}$ to conclude that $|N(B\mathbb{N})|$ is homotopy equivalent to a circle.

Remark. More generally, if M is a *commutative* monoid with group completion G, one can use Theorem A and Exercise 2(a) below to show that their classifying spaces are homotopy equivalent (in particular, the classifying space of a commutative monoid is 1-truncated). However, this is not true for a general monoid M.

Exercise 2. Recall that a category $\mathcal{C}$ is *filtered* if, for every finite category $\mathcal{I}$, every functor $f\colon \mathcal{I} \to \mathcal{C}$ extends to $\mathcal{I}^{\triangleright}$ (the category obtained from $\mathcal{I}$ by formally adding a final object). We can rephrase this condition by saying that the category of "cones under f"

$$\mathcal{C}_{f/} = \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \times_{\mathrm{Fun}(\mathcal{I},\mathcal{C})} \{f\}$$

is nonempty. A category is called *weakly contractible* if its nerve is weakly contractible.

(a) Show that every filtered category $\mathcal{C}$ is weakly contractible.

Hint. Write $\mathcal{C}$ as a filtered colimit of categories with final objects, and use the fact that the functors $\pi_n\colon \mathrm{sSet}_* \to \mathrm{Set}$ preserve filtered colimits.

(b) Deduce that the category $\mathcal{C}_{f/}$ is in fact weakly contractible for every functor $f\colon \mathcal{I} \to \mathcal{C}$ from a finite category to a filtered category.

A category $\mathcal{C}$ is called *sifted* if it is nonempty and $\mathcal{C}_{f/}$ is weakly contractible for every functor $f\colon \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a discrete category with two objects. By (b), every filtered category is sifted.

(c) Show that every sifted category $\mathcal{C}$ is weakly contractible.

Hint. Apply Theorem A to the diagonal functor $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

(d) Show that Δ^{op} is sifted but not filtered.

Hint. Use an adjunction to reduce to the following the statement: for every $I, J \in \Delta$, the poset $\mathrm{sd}(I \times J)$ of totally ordered subsets of $I \times J$ is weakly contractible. To prove the latter, it may help to prove more generally that $\mathrm{sd}(P)$ is weakly contractible for every poset P with a final object.

Exercise 3. Let $\mathcal{C}$ be an exact category and $\mathcal{B} \subset \mathcal{C}$ a full subcategory containing 0 and closed under extensions (with the induced exact structure). Suppose that, for every $X \in \mathcal{C}$, there exists $X' \in \mathcal{C}$ such that $X \oplus X' \in \mathcal{B}$ (one says that $\mathcal{B}$ is *cofinal* in $\mathcal{C}$). Show that the induced map $K_0(\mathcal{B}) \to K_0(\mathcal{C})$ is injective.

Remark. The *cofinality theorem* states that furthermore $K_n(\mathcal{B}) \simeq K_n(\mathcal{C})$ for all $n \geq 1$.