SoSe 25 Algebraic K-theory. Exercise sheet 12

Exercise 1. Let $\mathcal{C}$ be an exact category. A chain complex (C_*, d) in $\mathcal{C}$ is called *exact* if each differential $d\colon C_{i+1} \to C_i$ factors as $C_{i+1} \twoheadrightarrow C'_i \rightarrowtail C_i$ and $C'_i \rightarrowtail C_i \twoheadrightarrow C'_{i-1}$ is an exact sequence.

Let $\mathcal{C}$ and $\mathcal{D}$ be exact categories and let

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

be a chain complex of exact functors $\mathcal{C} \to \mathcal{D}$ which is objectwise exact in the above sense. Show that there is a null-homotopy

$$\sum_{i=0}^{n} (-1)^i (F_i)_* \simeq 0\colon K(\mathcal{C}) \to K(\mathcal{D}).$$

Hint. First show that the functors F'_i are exact, then apply the additivity theorem.

Exercise 2. Let R be a commutative ring and let $i\colon P \to Q$ be a map in $\mathrm{Proj}(R)$. Show that the following conditions are equivalent:

(a) i is an admissible monomorphism for the minimal exact structure, i.e., it identifies P with a direct summand of Q.

(b) The dual map $i^\vee\colon Q^\vee \to P^\vee$ is surjective.

(c) i is universally injective, i.e., for every map of commutative rings $R \to S$, $i \otimes_R S\colon P \otimes_R S \to Q \otimes_R S$ is injective.

(d) For every $x \in \mathrm{Spec}(R)$, $i \otimes_R \kappa(x)$ is injective.

Remark. These characterizations of admissible monomorphisms in $\mathrm{Proj}(R)$ hold more generally for $\mathrm{Vect}(X)$ for any scheme X.

Exercise 3. Let $\mathcal{C}$ be an exact category and $\mathcal{P} \subset \mathcal{C}$ a full subcategory containing 0 and closed under extensions. Suppose that:

(1) for every exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{C}$, if $Y, Z \in \mathcal{P}$, then $X \in \mathcal{P}$;

(2) for every $X \in \mathcal{C}$, there exists an admissible epimorphism $P \twoheadrightarrow X$ with $P \in \mathcal{P}$.

Let $\mathcal{P}_n \subset \mathcal{C}$ be the full subcategory of objects having a $\mathcal{P}$-resolution of length $\leq n$. Prove the following statements for every $n \geq 0$:

(a) $\mathcal{P}_n$ is closed under extensions in $\mathcal{C}$.

(b) If $X \rightarrowtail Y \twoheadrightarrow Z$ is an exact sequence in $\mathcal{C}$ with $Y \in \mathcal{P}_n$ and $Z \in \mathcal{P}_{n+1}$, then $X \in \mathcal{P}_n$.

Exercise 4. Let X be a noetherian scheme, $i\colon Z \hookrightarrow X$ a closed immersion, and $j\colon U \hookrightarrow X$ the complementary open immersion. Let $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$ be the full subcategory of coherent sheaves $\mathcal{F}$ such that $j^*(\mathcal{F}) = 0$. Prove the following statements:

(a) The functor $i_*\colon \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$ is fully faithful and restricts to a functor $i_*\colon \mathrm{Coh}(Z) \to \mathrm{Coh}_Z(X)$.

(b) For every sheaf $\mathcal{F} \in \mathrm{Coh}_Z(X)$, there exists a finite filtration

$$0 = \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{F}$$

such that $\mathcal{F}_{k-1}/\mathcal{F}_k$ is in the essential image of $i_*\colon \mathrm{Coh}(Z) \to \mathrm{Coh}_Z(X)$.

Hint. Consider the filtration $\mathcal{I}^k\mathcal{F}$ where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal of Z.