SoSe 25 Algebraic K-theory. Exercise sheet 13

Exercise 1. A full subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if it contains 0 and is closed under subobjects, quotients, and extensions. In this situation, $\mathcal{B}$ is an abelian category and the quotient $\mathcal{A}/\mathcal{B}$ exists in the 2-category of abelian categories and exact functors. The category $\mathcal{A}/\mathcal{B}$ has the same objects as $\mathcal{A}$ and

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y) = \operatorname*{colim}_{X' \rightarrowtail X, Y \twoheadrightarrow Y'} \mathrm{Hom}_{\mathcal{A}}(X',Y'),$$

where the colimit is taken over all $\mathcal{B}$-admissible monos $X' \rightarrowtail X$ and epis $Y \twoheadrightarrow Y'$ (i.e., whose (co)kernels are in $\mathcal{B}$).

Let R be a noetherian ring and $f \in R$. Let $\mathrm{Coh}(R)_{f\text{-nil}} \subset \mathrm{Coh}(R)$ be the full subcategory of f-power torsion coherent R-modules. Show that $\mathrm{Coh}(R)_{f\text{-nil}}$ is a Serre subcategory of $\mathrm{Coh}(R)$ and that the localization functor $\mathrm{Coh}(R) \to \mathrm{Coh}(R[\frac{1}{f}])$, $M \mapsto M[\frac{1}{f}]$, induces an equivalence of categories

$$\mathrm{Coh}(R)/\mathrm{Coh}(R)_{f\text{-nil}} \simeq \mathrm{Coh}(R[\tfrac{1}{f}]).$$

Hint. To prove essential surjectivity, use that every R-module is the union of its coherent submodules.

Remark. More generally, we have $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \simeq \mathrm{Coh}(X - Z)$ for any noetherian scheme X and closed subset $Z \subset X$.

Exercise 2. Let R be a noetherian ring and let $i\colon \mathrm{Spec}(R) \to \mathbb{A}^1_R$ be the zero section. Using the additivity theorem, show that the map

$$i_*\colon G(R) \to G(R[t])$$

is null-homotopic.

Exercise 3. Let D be a Dedekind domain with fraction field F. Show that there is a long exact sequence

$$\cdots \to K_{i+1}(F) \to \bigoplus_{\mathfrak{m} \subset D} K_i(D/\mathfrak{m}) \to K_i(D) \to K_i(F) \to \cdots \to K_0(F) \to 0,$$

where $\mathfrak{m}$ ranges over the maximal ideals in D.

Hint. Write F as the union of the subrings $D[\frac{1}{f}]$ with $f \in D - \{0\}$.