SoSe 26 Algebraic Geometry II
Exercise sheet 1 (due April 23)

Exercise 1.1. (2 points) Let X be a noetherian scheme.

(a) Show that every open subscheme $U \subset X$ is noetherian.

(b) Show that every closed subscheme $Z \subset X$ is noetherian.

Exercise 1.2. (6 points) A ring R is called *coherent* if $\mathrm{Mod}_R^{\mathrm{fp}}$ is an abelian category such that $\mathrm{Mod}_R^{\mathrm{fp}} \hookrightarrow \mathrm{Mod}_R$ is exact.

(a) Show that R is coherent if and only if any finitely generated ideal in R is finitely presented (as an R-module).

Show that "coherent" is a local property of rings, i.e.:

(b) If R is coherent and $f \in R$, then R_f is coherent.

(c) If $(f_1, \ldots, f_n) = R$ and each R_{f_i} is coherent, then R is coherent.

Exercise 1.3. (3 points) Let T be a topological space. Prove the following statements:

(a) Let K be an irreducible component of T and let $U \subset T$ be open. Then $U \cap K$ is either empty or an irreducible component of U.

(b) Let $(U_i \subset T)_{i \in I}$ be an open covering of T such that $I \neq \varnothing$ and $U_i \cap U_j \neq \varnothing$ for all $i, j \in I$. Then T is irreducible if and only if each U_i is irreducible.

Exercise 1.4. (3 points)

(a) Let X be an integral scheme with generic point η. Show that, for every nonempty open subscheme $U \subset X$, the restriction map $\mathcal{O}(U) \to \kappa(\eta)$ is injective. In particular, for any nonempty opens $V \subset U$, the restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is injective.

Hint. Recall that if $x \in X$ and U is an affine open containing x, then $\kappa(x) \simeq \kappa(\mathfrak{p})$ where $\mathfrak{p} \subset \mathcal{O}(U)$ is the prime ideal corresponding to x.

(b) Find an example of an irreducible scheme X and a nonempty open $U \subset X$ such that $\mathcal{O}(X) \to \mathcal{O}(U)$ is not injective.