

SoSe 26 ALGEBRAIC GEOMETRY II
EXERCISE SHEET 10 (DUE JULY 9)

Exercise 10.1. (2 points) Let T be a topological space and A a commutative monoid. Show that the presheaf of cycles

$$\mathcal{Z}^*(-, A): \text{Open}(T)^{\text{op}} \rightarrow \text{Set}$$

is a sheaf on T .

Exercise 10.2. (8 points) Let X be a locally noetherian scheme. Let div' be the map

$$\mathcal{K}(X)^\times \xrightarrow{\text{div}'} \text{Div}(X) \xrightarrow{\text{cyc}} \mathcal{Z}^1(|X|, \mathbb{Z}).$$

The *Weil divisor class group* $\text{Cl}(X)$ is the cokernel of div' .

Consider the projection $p: \mathbb{A}_X^1 \rightarrow X$. If $x \in |\mathbb{A}_X^1|$ is a point of codimension 1, its image $p(x)$ in $|X|$ has codimension either 0 or 1, since p is flat. This defines a decomposition

$$\mathcal{Z}^1(|\mathbb{A}_X^1|, \mathbb{Z}) = \mathcal{Z}_{(0)}^1 \oplus \mathcal{Z}_{(1)}^1.$$

Taking preimages of irreducible closed subsets induces maps

$$p^*: \mathcal{Z}^1(|X|, \mathbb{Z}) \rightarrow \mathcal{Z}^1(|\mathbb{A}_X^1|, \mathbb{Z}) \quad \text{and} \quad p^*: \text{Cl}(X) \rightarrow \text{Cl}(\mathbb{A}_X^1).$$

The goal of this problem is to show that the latter is an isomorphism when X is *reduced*. For simplicity, suppose that X is integral with function field K .

- (a) Define isomorphisms $\mathcal{Z}_{(0)}^1 \simeq \mathcal{Z}^1(|\mathbb{A}_K^1|, \mathbb{Z})$ and $\mathcal{Z}_{(1)}^1 \simeq \mathcal{Z}^1(|X|, \mathbb{Z})$.
- (b) Show that the commutative square

$$\begin{array}{ccc} K^\times & \xrightarrow{\text{div}'} & \mathcal{Z}^1(|X|, \mathbb{Z}) \\ \downarrow & & \downarrow p^* \\ K(t)^\times & \xrightarrow{\text{div}'} & \mathcal{Z}^1(|\mathbb{A}_X^1|, \mathbb{Z}) \end{array}$$

is cartesian, and hence that $p^*: \text{Cl}(X) \rightarrow \text{Cl}(\mathbb{A}_X^1)$ is injective.

- (c) Let $x \in |\mathbb{A}_X^1|$ be a point of codimension 1 such that $p(x)$ has codimension 0. Find a rational function f on \mathbb{A}_X^1 such that $x - \text{div}'(f) \in \mathcal{Z}_{(1)}^1$.

Hint. $K[t]$ is a principal ideal domain.

- (d) Deduce that $p^*: \text{Cl}(X) \rightarrow \text{Cl}(\mathbb{A}_X^1)$ is an isomorphism.
- (e) Suppose that the local rings of X are unique factorization domains (for example, X is regular). Deduce that $p^*: \text{Pic}(X) \rightarrow \text{Pic}(\mathbb{A}_X^1)$ is an isomorphism.

Remark. If X is only assumed normal, there is an injective map $\text{Pic}(X) \hookrightarrow \text{Cl}(X)$ whose image consists of classes of Weil divisors that are locally in the image of div' (since $\text{div}: \mathcal{K}^\times \rightarrow \text{Div}$ is an epimorphism of sheaves). One can still deduce that $p^*: \text{Pic}(X) \rightarrow \text{Pic}(\mathbb{A}_X^1)$ is an isomorphism by showing that $p^*: \mathcal{Z}^1(|X|, \mathbb{Z}) \rightarrow \mathcal{Z}^1(|\mathbb{A}_X^1|, \mathbb{Z})$ detects such Weil divisors.

Exercise 10.3. (4 points) Let X be a scheme. Prove that the following conditions are equivalent:

- (a) $X \simeq \text{Spec}(R)$, where R is a domain in which the divisibility relation is total.
- (b) X is quasi-compact, reduced, and its poset of closed subschemes is totally ordered.
- (c) X has no nontrivial open coverings (i.e., every open covering contains X) and no nontrivial blowups at finitely presented closed subschemes (i.e., for any finitely presented $Z \subsetneq X$, $\text{Bl}_Z(X) \xrightarrow{\sim} X$).

Such a ring R is called a *valuation ring*.