SoSe 26 Algebraic Geometry II
Exercise sheet 11 (due July 16)

Exercise 11.1. (2 points) Let R be a Prüfer domain, i.e., a domain in which every nonzero finitely generated ideal is a line. Show that R is integrally closed.

Hint. Let $r/s \in \mathrm{Frac}(R)$. By definition, (r,s) is a line. Choose $(f_1,\dots,f_n) = R$ such that (r,s) is principal in R_{f_i}, so that the fraction can be simplified. Assuming an integral equation for r/s over R, deduce that r/s belongs to R_{f_i} for all i, and hence to R.

Exercise 11.2. (6 points) Show that prim morphisms have the right lifting property with respect to the following maps:

(a) The open immersion $C - x \hookrightarrow C$, where C is a Dedekind scheme and $x \in |C|$ is a codimension 1 point (e.g., $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$ or $\mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ for a field k).

Hint. By the valuative criterion, we can lift from $\mathrm{Spec}(\mathcal{O}_{C,x}) - x$ to $\mathrm{Spec}(\mathcal{O}_{C,x})$. Then use the fact that prim morphisms are locally of finite type to extend this lift to an open neighborhood of x in C, and hence to all of C.

(b) The inclusion $\mathrm{Spec}(K) \hookrightarrow \mathrm{Spec}(R)$, where R is a Dedekind domain with fraction field K (e.g., $\mathrm{Spec}(\mathbb{Q}) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$).

Hint. First lift to some open subscheme of $\mathrm{Spec}(R)$, then use (a).

Exercise 11.3. (4 points) Let X be a scheme.

(a) Let $D \subset X$ be an effective Cartier divisor. Show that $|X| - |D|$ is dense in $|X|$.

Hint. Recall that $\mathrm{Spec}(\varphi)$ is dominant if and only if $\ker(\varphi)$ is a nilideal.

(b) Let $Z \subset X$ be a closed subscheme of finite presentation. Show that the blowup $\mathrm{Bl}_Z(X) \to X$ is surjective if and only if $|X| - |Z|$ is dense in $|X|$.