SoSe 26 Algebraic Geometry II
Exercise sheet 2 (due April 30)

Exercise 2.1. (2 points) Let X be a scheme whose local rings are integral domains. Show that the irreducible components of $|X|$ are pairwise disjoint.

Exercise 2.2. (2 points) Let X be a scheme and let $x \in |X|$. Prove the following statements:

(a) For any open subscheme $U \subset X$ containing x, $\operatorname{codim}(x, X) = \operatorname{codim}(x, U)$.

(b) $\operatorname{codim}(x, X) = \dim(\mathcal{O}_{X,x})$.

Exercise 2.3. (8 points) Let k be a field and let $N = \operatorname{Spec}(k[x, y]/(y^2 - x^3 - x^2))$ be the affine nodal cubic over k.

(a) Find a system of parameters of the local ring of N at $(0, 0) \in N(k)$.

(b) Find a basis of the Zariski cotangent space $\mathrm{T}^*_{(0,0)}(N)$.

Fix $(a, b) \in N(k)$ with $(a, b) \neq (0, 0)$.

(c) Find a basis of the Zariski cotangent space $\mathrm{T}^*_{(a,b)}(N)$.

(d) Compute the k-linear embedding $\mathrm{T}_{(a,b)}(N) \hookrightarrow \mathrm{T}_{(a,b)}(\mathbb{A}^2_k) \simeq k^2$ induced by the closed immersion $N \hookrightarrow \mathbb{A}^2_k$.

Exercise 2.4. (6 points) Let k be a field of characteristic $\neq 2$ and consider the surface $S = \operatorname{Spec}(R)$ where $R = k[x, y, z]/(xy - z^2)$.

(a) Identify R with the fixed points of the C_2-action on the k-algebra $k[u, v]$ given by $u \mapsto -u$ and $v \mapsto -v$.

(b) Deduce from (a) that R is an integrally closed domain.

Hint. Note that $k[u, v]^{\mathrm{C}_2} = k(u, v)^{\mathrm{C}_2} \cap k[u, v]$, and recall that $k[u, v]$ is a unique factorization domain and hence integrally closed.

(c) Find a basis of the Zariski cotangent space $\mathrm{T}^*_{(0,0,0)}(S)$ and deduce that S is not regular.