SoSe 26 Algebraic Geometry II
Exercise sheet 3 (due May 7)

Exercise 3.1. (4 points) Recall that a map of schemes $f\colon Y \to X$ is *quasi-compact* if the preimage by f of any quasi-compact open subscheme of X is quasi-compact, and it is *quasi-separated* if its diagonal $\Delta_f\colon Y \to Y \times_X Y$ is quasi-compact.

(a) Show that "quasi-compact" is local on the base.

(b) Deduce that "quasi-separated" is local on the base.

Remark. In the same way, the fact that "closed immersion" is local on the base implies that "separated" is local on the base.

Exercise 3.2. (5 points) Let X be a scheme and let $f \in \mathcal{O}(X)$.

(a) Show that the canonical map $\mathcal{O}(X)_f \to \mathcal{O}(\mathrm{D}(f))$ is injective if X is quasi-compact and bijective if X is qcqs.

Hint. Write X as a colimit of affine schemes.

(b) Let $X \to \mathrm{Spec}(R)$ be a morphism and let $f_1, \ldots, f_n \in R$ generate the unit ideal. Suppose that $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_{f_i})$ is affine for all i. Show that X is affine.

Hint. By Exercise 3.1, X is qcqs. Use (a) to show that the canonical map $X \to \mathrm{Spec}(\mathcal{O}(X))$ is an isomorphism.

(c) Deduce from (b) that "affine" is local on the base. (Recall that $Y \to X$ is affine if and only if, for every ring R and every map $\mathrm{Spec}(R) \to X$, $Y \times_X \mathrm{Spec}(R)$ is affine.)

Exercise 3.3. (2 points) Let $R \to S$ be a ring map. Construct a functor

$$((\mathrm{CAlg}_R)_{/S})^{\mathrm{op}} \times \mathrm{Mod}_S \to \mathrm{Mod}_S$$

sending (S', M) to $\mathrm{Der}_R(S', M)$.

Exercise 3.4. (4 points) Let R be a ring, let $P = R[x_1, \ldots, x_n]$, and let $\mathrm{d}\colon P \to \Omega_{P/R}$ be the universal R-derivation on P.

(a) Show that $\Omega_{P/R}$ is a free P-module with basis $(\mathrm{d}x_1, \ldots, \mathrm{d}x_n)$.

(b) Let $Q = R[y_1, \ldots, y_m]$ and let $f\colon Q \to P$ be an R-algebra map, given by m polynomials in n variables $(f_1, \ldots, f_m)$. Show that the P-linear map $\Omega_{Q/R} \otimes_Q P \to \Omega_{P/R}$ induced by the R-derivation $\mathrm{d} \circ f\colon Q \to \Omega_{P/R}$ is given by the Jacobian matrix $(\partial f_j/\partial x_i)_{i,j}$.