SoSe 26 Algebraic Geometry II
Exercise sheet 4 (due May 21)

Exercise 4.1. (8 points) Let k be a ring. Compute the conormal modules of the following ring maps:

(a) $k \hookrightarrow k[x]/(x^2)$.

(b) $k \hookrightarrow k[x,y]/(x,y)^2$.

(c) $k[x,y]/(y^2 - x^3) \hookrightarrow k[t]$, $x \mapsto t^2$, $y \mapsto t^3$.

(d) $k[x,y]/(y^2 - x^3 - x^2) \hookrightarrow k[t]$, $x \mapsto t^2 - 1$, $y \mapsto t(t^2 - 1)$.

Exercise 4.2. (4 points) Let R be a ring, let $S = R[x_1, \ldots, x_n]/I$ with $I = (f_1, \ldots, f_m)$, and let $J = (\partial f_i/\partial x_j)_{i,j} \in \mathrm{Mat}_{m\times n}(R[x_1, \ldots, x_n])$. Recall the fundamental exact sequence of S-modules

$$0 \to \mathcal{N}_{S/R} \to I/I^2 \xrightarrow{\mathrm{d}} \bigoplus_{j=1}^{n} S\mathrm{d}x_j \to \Omega_{S/R} \to 0.$$

(a) Let $g \in S$ be such that the image of J in $\mathrm{Mat}_{m\times n}(S_{g_i})$ contains an invertible $m \times m$ submatrix. Show that the localization at g of the above sequence looks as follows:

$$0 \to 0 \to S_g^m \to S_g^n \to S_g^{n-m} \to 0.$$

(b) Suppose that the image of J in $\mathrm{Mat}_{m\times n}(\kappa(\mathfrak{m}))$ has rank m for every maximal ideal $\mathfrak{m} \subset S$. Show that $\Omega_{S/R}$ is a vector space of rank $n - m$ and that $\mathcal{N}_{S/R} = 0$.

Exercise 4.3. (3 points) Let X be an algebraic functor, let $M \in \mathrm{Mod}_X$, and let $p\colon \mathbb{A}(M) \to X$ be the associated affine space. Recall that the R-points of $\mathbb{A}(M)$ are pairs (x, a) with $x \in X(R)$ and $a\colon M(x) \to R$ an R-linear map. Show that $\Omega_p \simeq p^*(M)$.

Exercise 4.4. (9 points) Let $\varphi\colon R \twoheadrightarrow \bar{R}$ be a surjective ring map and consider the base change functor

$$\varphi^*\colon \mathrm{Vect}_R \to \mathrm{Vect}_{\bar{R}}, \quad V \mapsto \bar{V}.$$

Let $V, W \in \mathrm{Vect}_R$.

(a) Show that the map $\varphi^*\colon \mathrm{Map}(V, W) \to \mathrm{Map}(\bar{V}, \bar{W})$ is surjective.

(b) Suppose that φ is local (i.e., detects units). Show that the map from (a) detects isomorphisms.

Hint. First assume $V = W$, and deduce the general case using Zariski descent.

Suppose now that φ is a thickening, i.e., that its kernel consists of nilpotent elements.

(c) Show that φ is local.

(d) Show that φ induces a bijection $\mathrm{Idem}(R) \xrightarrow{\sim} \mathrm{Idem}(\bar{R})$.

Hint. Recall that $e \mapsto \mathrm{D}(e)$ is a bijection $\mathrm{Idem}(R) \xrightarrow{\sim} \mathrm{Clopen}(\mathrm{Spec}(R))$.

(e) Let $n \in \mathbb{N}$. Show that every idempotent endomorphism $\bar{e}$ of $\bar{R}^n$ lifts to an idempotent endomorphism of R^n.

Hint. Let $e \in \mathrm{End}_R(R^n)$ be any lift of $\bar{e}$ and let $\chi_e \in R[x]$ be its characteristic polynomial. The theorem of Cayley–Hamilton asserts that the kernel I of the map $R[x] \to \mathrm{End}_R(R^n)$, $x \mapsto e$, satisfies $I^n \subset (\chi_e) \subset I$. Let A be the image of this map, and define $\bar{A}$ similarly. Deduce that $A \twoheadrightarrow \bar{A}$ is a thickening and apply (d).

(f) Conclude that $\varphi^* \colon \mathrm{Vect}_R \to \mathrm{Vect}_{\bar{R}}$ induces a bijection between isomorphism classes.