SoSe 26 Algebraic Geometry II
Exercise sheet 5 (due May 28)

Exercise 5.1. (3 points) Let $f\colon Y \to X$ be a map of schemes. Prove that f is smooth if and only if, for every $y \in |Y|$, there exist affine open subschemes $V \subset Y$ and $U \subset X$ with $y \in |V|$ and $f(V) \subset U$ such that $f|_V$ factors as

$$V \to \mathbb{A}^n_U \to U,$$

where $V \to \mathbb{A}^n_U$ is étale.

Hint. Use the Jacobian criterion directly.

Exercise 5.2. (6 points) Let $f\colon Y \to X$ be a map of schemes which is locally of finite type. Prove the equivalence of the following conditions:

(a) f is unramified.

(b) $\Omega_f = 0$.

(c) For every algebraically closed field k and every $x\colon \mathrm{Spec}(k) \to X$, there exist a set I and an isomorphism of k-schemes $Y_x \simeq \coprod_{i\in I} \mathrm{Spec}(k)$.

Hint. Prove (b) $\Rightarrow$ (a) using the fundamental exact sequence. For (a) $\Rightarrow$ (c), show that unramified schemes over a field are étale. For (c) $\Rightarrow$ (b), use Nakayama's lemma.

Exercise 5.3. (2 points) Let $f\colon X \to Z$ and $g\colon Y \to Z$ be maps of algebraic functors admitting cotangent modules. Show that the product map $h = f \times_Z g\colon X \times_Z Y \to Z$ admits a cotangent module, which is given by

$$\Omega_h(x, y) = \Omega_f(x) \oplus \Omega_g(y).$$

Exercise 5.4. (2 points) Let k be a ring and M a projective k-module. Show that $\mathbb{P}(M) \to \mathrm{Spec}(k)$ has the right lifting property with respect to thickenings and hence is formally smooth.

Hint. Let $R \twoheadrightarrow \bar{R}$ be a thickening where R is a k-algebra, and let $\bar{a}\colon M \otimes_k \bar{R} \twoheadrightarrow \bar{L}$ be a quotient line. By Exercise 4.4, we can lift $\bar{L}$ to a line L over R. Pick an arbitrary R-linear map $a\colon M \otimes_k R \to L$ lifting $\bar{a}$, and show that a is surjective by considering its epimorphism locus.

Exercise 5.5. (3 points) Consider a cartesian square of rings

$$\begin{array}{ccc} A \times_C B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

where $A \to C \leftarrow B$ are surjective.

(a) Show that the induced square of prime spectra is a cocartesian square in the category of ringed spaces.

(b) Deduce that schemes are cohesive, i.e., send the above square to a cartesian square.

Hint. Recall that $X \mapsto (|X|, \mathcal{O}_X)$ defines a fully faithful embedding $\mathrm{Sch} \hookrightarrow \mathrm{LRS}$ and the the (non-full) inclusion $\mathrm{LRS} \hookrightarrow \mathrm{RS}$ preserves colimits.