# SoSe 26 Algebraic Geometry II
# Exercise sheet 6 (due June 11)

**Exercise 6.1.** (4 points) Let $k$ be a ring, $M$ a $k$-module, and $n \in \mathbb{N}$. Let $f\colon \mathrm{Gr}_n(M) \to \mathrm{Spec}(k)$ be the canonical map and let $f^*(M) \twoheadrightarrow \mathcal{T}$ be the universal rank $n$ quotient in $\mathrm{Mod}_{\mathrm{Gr}_n(M)}$. Show that the cotangent module of the Grassmannian is computed by a short exact sequence

$$0 \to \Omega_f \to f^*(M) \otimes \mathcal{T}^\vee \to \mathcal{T} \otimes \mathcal{T}^\vee \to 0.$$

**Exercise 6.2.** (8 points) Two $R$-modules $M$ and $N$ are called *Tor-independent* over $R$ if $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. A commutative square of rings

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R' & \longrightarrow & S' \end{array}$$

is called *Tor-independent* if it is cocartesian and $S$ and $R'$ are Tor-independent $R$-modules.

(a) Let $R \to S$ be a ring map, $M$ an $R$-module, and $N$ an $S$-module. Suppose that $M$ and $S$ are Tor-independent over $R$ and that $M \otimes_R S$ and $N$ are Tor-independent over $S$. Show that $M$ and $N$ are Tor-independent over $R$.

Deduce that the collection of Tor-independent squares is closed under pasting.

*Hint.* Recall that $\mathrm{Tor}_i^R(M, -)$ can be computed as $\mathrm{H}_i(P_* \otimes_R -)$ where $P_*$ is a projective resolution of $M$.

A family $(f_\alpha)_{\alpha \in A}$ in a ring $R$ is called *regular* if $R$ and $\mathbb{Z}$ are Tor-independent over the polynomial ring $\mathbb{Z}[t_A] = \mathbb{Z}[t_\alpha \mid \alpha \in A]$, where $t_\alpha$ acts as $f_\alpha$ on $R$ and as $0$ on $\mathbb{Z}$.

(b) Let $f_1, \ldots, f_n \in R$ be such that, for each $i$, $f_i$ is not a zero divisor in $R/(f_1, \ldots, f_{i-1})$. Show that $(f_1, \ldots, f_n)$ is a regular family in $R$.

*Hint.* Use induction on $n$ and (a).

Suppose now that $(f_\alpha)_{\alpha \in A}$ is a regular family in $R$. Let $I = (f_\alpha \mid \alpha \in A) \subset R$ and let $J = (t_\alpha \mid \alpha \in A) \subset \mathbb{Z}[t_A]$.

(c) Show that $R$ and $\mathbb{Z}[t_A]/J^n$ are Tor-independent over $\mathbb{Z}[t_A]$ for all $n \geq 0$.

*Hint.* Use induction on $n$. Observe that the $\mathbb{Z}[t_A]$-module $J^n/J^{n+1}$ is a direct sum of copies of $\mathbb{Z}[t_A]/J = \mathbb{Z}$.

(d) Deduce that the canonical map $J^n \otimes_{\mathbb{Z}[t_A]} R \to I^n$ is an isomorphism for all $n \in \mathbb{N}$.

(e) Deduce that the canonical map $\mathrm{Sym}_R^n(I/I^2) \to I^n/I^{n+1}$ is an isomorphism for all $n \in \mathbb{N}$.

*Hint.* By (d), it suffices to prove this for $J$ instead of $I$.

*Remark.* Similarly, we have $\mathrm{Sym}_R^n(I) \xrightarrow{\sim} I^n$, as this can be shown for $J$.

**Exercise 6.3.** (4 points) Let $X$ be a scheme. A *cone* with vertex $X$ is a closed subscheme of some affine space $\mathbb{A}(M)$ with $M \in \mathrm{Mod}_X$ that is $(\mathbb{A}^1, \cdot)$-invariant, i.e., closed under scalar multiplication. Equivalently, it is Spec of a quasi-coherent $\mathbb{N}$-graded algebra over $X$ generated in degree 1.

Given a closed subscheme $Z \subset X$ with quasi-coherent ideal $I$, the *normal cone* of $Z$ in $X$ is the cone $\mathrm{Spec}(\mathrm{gr}_I \mathcal{O}_X)$ with vertex $Z$, where

$$\mathrm{gr}_I \mathcal{O}_X = \bigoplus_{n \in \mathbb{N}} I^n / I^{n+1} \in \mathrm{CAlg}^{\mathbb{N}}_{\mathcal{O}_X/I} \simeq \mathrm{CAlg}^{\mathbb{N}}_Z$$

is the associated graded of the $I$-adic filtration of $\mathcal{O}_X$. By Exercise 6.2(e), the normal cone is simply the affine space $\mathbb{A}(\mathcal{N}_{Z/X})$ when $I$ is locally generated by a regular family, but it is some subcone of it in general.

Compute and interpret geometrically the normal cone of $Z$ in $X$ in the following cases:

(a) $X = \mathrm{Spec}(k[x,y]/(y^2 - x^3 - x^2))$, $Z = \mathrm{V}(x, y)$.

(b) $X = \mathrm{Spec}(k[x,y]/(y^2 - x^3))$, $Z = \mathrm{V}(x, y)$.