SoSe 26 Algebraic Geometry II
Exercise sheet 7 (due June 18)

Exercise 7.1. (3 points) Let R be a ring, let $f \in R$, and let $Z = \mathrm{V}(f) \subset \mathrm{Spec}(R)$. Show that $\mathrm{vBl}_Z(X)$ and $\mathrm{Bl}_Z(X)$ are the closed subschemes of X defined by the ideals of f-torsion and f-power torsion elements, respectively.

Exercise 7.2. (5 points) Let X be an algebraic functor and let

$$\begin{array}{ccc} M & \overset{r}{\twoheadrightarrow} & N \\ & {\scriptstyle z}\searrow \quad \swarrow{\scriptstyle w} & \\ & \mathcal{O}_X & \end{array}$$

be a commutative triangle in Mod_X, where r is an epimorphism.

(a) Show that the induced map $\mathrm{vBl}_w(X) \to \mathrm{vBl}_z(X)$ is a closed immersion.

Suppose now that X is a scheme. Show that the induced map $\mathrm{Bl}_w(X) \to \mathrm{Bl}_z(X)$ is an *isomorphism* in two ways:

(b) by observing that the Cartier locus in $\mathrm{vBl}_z(X)$ is contained in $\mathrm{vBl}_w(X)$;

(c) by using that $\mathrm{Bl}_z(X)$ is the closure of $\mathrm{D}(z)$ in $\mathrm{vBl}_z(X)$.

In particular, taking N to be image of z, we see that $\mathrm{Bl}_z(X)$ depends only on the vanishing locus of z: $\mathrm{Bl}_z(X) = \mathrm{Bl}_{\mathrm{V}(z)}(X)$.

(d) Deduce that, if the exceptional divisor on $\mathrm{vBl}_z(X)$ happens to be non-virtual, then $\mathrm{Bl}_{\mathrm{V}(z)}(X) = \mathrm{vBl}_z(X)$.

Exercise 7.3. (6 points) Let $n \geq 0$.

(a) Let $z\colon \mathcal{O}_{\mathbb{A}^n}^n \to \mathcal{O}_{\mathbb{A}^n}$ be the tautological map in $\mathrm{Mod}_{\mathbb{A}^n}$. Define an isomorphism

$$\mathrm{vBl}_z(\mathbb{A}^n) \simeq \mathbb{A}_{\mathbb{P}^{n-1}}(\mathcal{O}(1)).$$

(b) Use Exercise 7.2(d) to show that $\mathrm{Bl}_0(\mathbb{A}^n) \simeq \mathbb{A}_{\mathbb{P}^{n-1}}(\mathcal{O}(1))$.

(c) Show that $\mathrm{Bl}_0(\mathbb{P}^n) \simeq \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}(1) \oplus \mathcal{O})$.

Hint. Define a map $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{P}^n$ and use (b) to show that it is the blowup at the origin. Alternatively, repeat (a) with $z\colon \mathcal{O}_{\mathbb{P}^n}(-1)^n \to \mathcal{O}_{\mathbb{P}^n}$.