SoSe 26 Algebraic Geometry II
Exercise sheet 8 (due June 25)

Exercise 8.1. (4 points) Let k be a ring and let $N = \mathrm{Spec}(k[x,y]/(y^2 - x^3 - x^2)) \subset \mathbb{A}^2_k$.

(a) Compute the strict transform of N at 0, i.e., the closure of $N - 0$ in $\mathrm{Bl}_0(\mathbb{A}^2_k)$.

(b) If k is a field, show that $\mathrm{Bl}_0(N)$ is isomorphic over N to the normalization of N.

Exercise 8.2. (2 points) Let X be a scheme and $Z \subset X$ a closed subscheme. Prove the following statements:

(a) If X is reduced, then $\mathrm{Bl}_Z(X)$ is reduced.

(b) If X is integral and $Z \neq X$, then $\mathrm{Bl}_Z(X)$ is integral.

Exercise 8.3. (3 points) Let X be a scheme and let $M, N \in \mathrm{Mod}_X$. Show that the deformation space $\mathrm{Def}_{\mathbb{A}(N)}(\mathbb{A}(M \oplus N))$ is canonically isomorphic to $\mathbb{A}(M \oplus N) \times \mathbb{A}^1$.

Hint. This can be proved directly from the definition, or by using the description of the deformation space in terms of blowups.

Exercise 8.4. (6 points)

(a) Let X be a scheme and $(U_i \subset X)_{i \in I}$ an open covering. Construct an equivalence between:

- the category of line bundles L over X such that $L|_{U_i}$ is trivial for each i;
- a certain category whose objects are families of nonvanishing functions $u_{ij} \in \mathcal{O}^\times(U_i \cap U_j)$ satisfying the cocycle condition $u_{ij}u_{jk} = u_{ik}$ in $\mathcal{O}^\times(U_i \cap U_j \cap U_k)$.

Hint. Use Zariski descent for quasi-coherent modules.

(b) Let R be a ring and let $n \in \mathbb{N}$. Suppose that every line over $R[x_1, \dots, x_n]$ is trivial (this holds for example if R is a unique factorization domain). Using (a), show that every line bundle over $\mathbb{P}^n_R$ is isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.