SoSe 26 Algebraic Geometry II
Exercise sheet 9 (due July 2)

Exercise 9.1. (3 points) Let X be a scheme, L a line bundle over X, and $s_0, \ldots, s_n \in \Gamma(X, L)$ global sections of L. Prove the following statements:

(a) There is a largest subfunctor $U \subset X$ for which the map

$$\varphi\colon U \to \mathbb{P}^n, \quad x \mapsto [s_0(x) : \ldots : s_n(x)],$$

is well-defined. Moreover, U is open in X.

(b) There is a closed subscheme $Z \subset X$ with $Z \cap U = \varnothing$ such that φ extends to a map $\mathrm{Bl}_Z(X) \to \mathbb{P}^n$.

Hint. It extends in fact to a suitable virtual blowup of X.

Exercise 9.2. (5 points) Show that the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the point $([1:0],[1:0])$ is isomorphic to the blowup of $\mathbb{P}^2$ at the two points $[1:0:0] \sqcup [0:1:0]$.

Hint. Call $x_1, y_1\colon \pi_1^*(\mathcal{O}(-1)) \to \mathcal{O}$ and $x_2, y_2\colon \pi_2^*(\mathcal{O}(-1)) \to \mathcal{O}$ the coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, and $x, y, z\colon \mathcal{O}(-1) \to \mathcal{O}$ the coordinates on $\mathbb{P}^2$. Show that these blowups agree with the virtual blowups $\mathrm{vBl}_{(y_1,y_2)}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathrm{vBl}_{(x,z)}\mathrm{vBl}_{(y,z)}(\mathbb{P}^2)$, then directly compare the R-points of these virtual blowups for any ring R.

Exercise 9.3. (5 points) Let R be a ring with the following property: every ideal I such that $\{r \in R \mid rI = 0\} = 0$ contains a non-zero divisor (it is well known that this holds when R is noetherian).

(a) Let $(f_1, \ldots, f_n) = R$. Show that the diagram of total rings of fractions

$$\mathrm{Frac}(R) \to \prod_i \mathrm{Frac}(R_{f_i}) \rightrightarrows \prod_{i,j} \mathrm{Frac}(R_{f_i f_j})$$

is an equalizer (recall that $\mathrm{Frac}(R) = R[S^{-1}]$ where $S \subset R$ is the set of non-zero divisors).

Hint. First reduce to the following statement: given $r_i, s_i \in R$ such that s_i is a non-zero divisor in R_{f_i} and $r_i s_j = r_j s_i$ in R, there exist $r, s \in R$ with s a non-zero divisor such that $r s_i = s r_i$ in R_{f_i}. To find s, apply the hypothesis to the ideal $I = \{s \in R \mid s_i \text{ divides } s r_i \text{ in } R_{f_i} \text{ for all } i\}$. Then find r using Zariski descent.

(b) Deduce that $\mathcal{K}(\mathrm{Spec}(R)) \simeq \mathrm{Frac}(R)$.