ALGEBRAIC GEOMETRY

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Contents

Date: May 1, 2026.

Conventions and notation.

- Throughout, "ring" means "commutative ring" and "algebra" means "commutative algebra". Not necessarily commutative algebras will be called "associative algebras".
- We denote by CAlg the category of (commutative) rings. Given $R \in \mathrm{CAlg}$, we denote by CAlg_R the category of (commutative) R-algebras.
- Given a ring R and a subset $S \subset R$, we denote by $R[S^{-1}]$ the localization of R at S. When $S = \{f\}$ has single element, we also write R_f or $R[\frac{1}{f}]$ for the localization.
- The words "morphism" and "map" are used interchangeably. We denote by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ the set of maps from X to Y in a category $\mathcal{C}$; we simply write $\mathrm{Map}(X, Y)$ if $\mathcal{C}$ is clear from the context.
- The symbols $\simeq$ and $\xrightarrow{\sim}$ are used for isomorphisms within a category as well as for equivalences of categories. The arrows $\hookrightarrow$ and $\twoheadrightarrow$ are sometimes used for monomorphisms and epimorphisms.
- Given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, we denote by $\mathcal{C}_{/X}$ the category of objects over X and by $\mathcal{C}_{X/}$ the category of objects under X. For example, $\mathrm{CAlg}_R \simeq \mathrm{CAlg}_{R/}$.
- $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category of functors from $\mathcal{C}$ to $\mathcal{D}$ (objects are functors, morphisms are natural transformations).
- If $F\colon \mathcal{C} \to \mathcal{D}$ is left adjoint to $G\colon \mathcal{D} \to \mathcal{C}$, we write $F \dashv G$. When writing the adjunction as
$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D},$$
the left adjoint is always on top.

1. Some examples

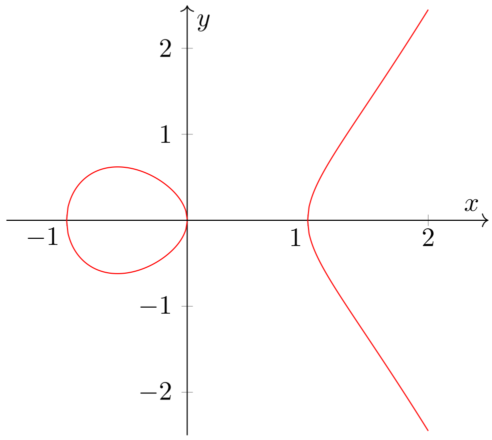
1.1. **Punctured elliptic curves.** Consider the polynomial equation in two variables $y^2 = x^3 - x$. We can consider its set of solutions in any ring R, namely

$$X(R) = \{(a, b) \in R^2 \mid b^2 = a^3 - a\}.$$

This defines a functor $X\colon \mathrm{CAlg} \to \mathrm{Set}$ from the category of rings to the category of sets, which is an example of an *affine scheme*. Being defined by a single equation in two variables, X is called a *family of algebraic curves* or simply a *curve*. It is also called an *arithmetic surface*: "arithmetic" because the equation has integral coefficients, and "surface" because it turns out to be a 2-dimensional object from the perspective of dimension theory in commutative algebra.

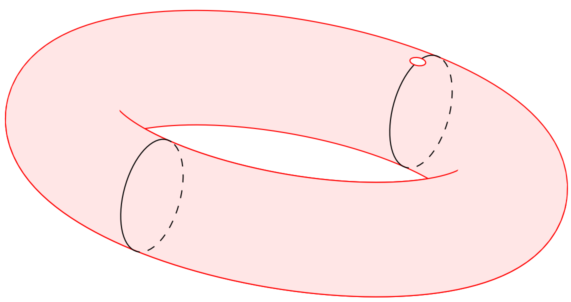
Let us consider explicitly the solution sets $X(R)$ for various rings R:

(i) $X(\mathbb{R})$ is a 1-dimensional real submanifold of $\mathbb{R}^2$, which is diffeomorphic to $S^1 \sqcup \mathbb{R}$:



(ii) For any subring $R \subset \mathbb{R}$, $X(R)$ is the subset of $X(\mathbb{R})$ consisting of all points whose coordinates lie in R. One can show for example that the only points of $X(\mathbb{R})$ with rational coordinates are those on the x-axis, so that $X(\mathbb{Q}) = X(\mathbb{Z}) = \{(0, 0), (\pm 1, 0)\}$.

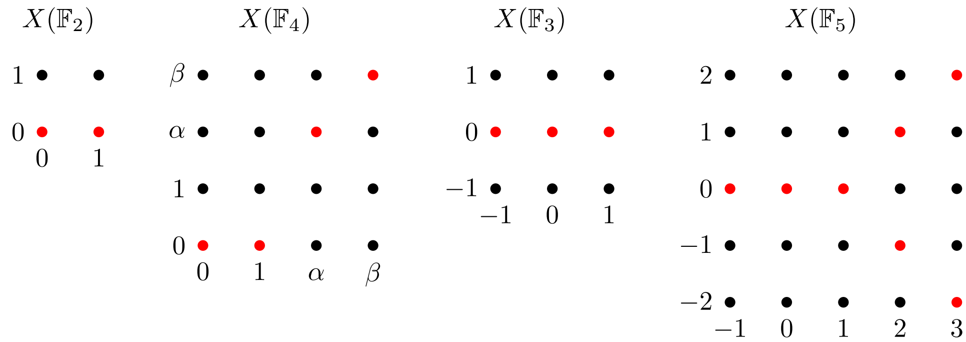
(iii) $X(\mathbb{C})$ is a 1-dimensional complex submanifold of $\mathbb{C}^2$, which can be shown to be diffeomorphic to a punctured torus:



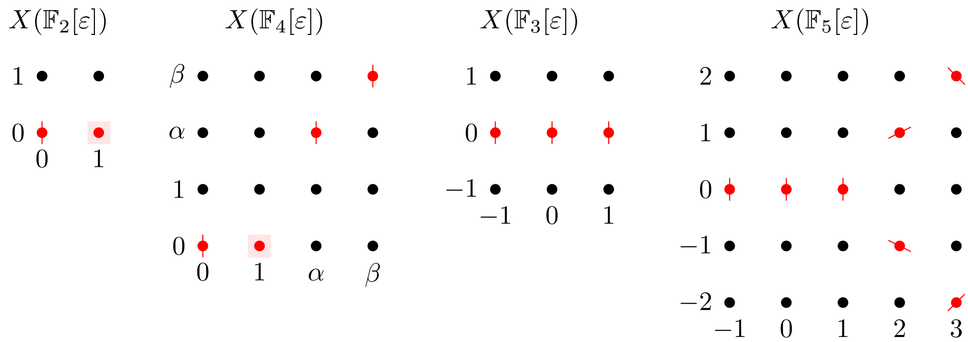
The two black circles indicate the intersection of $X(\mathbb{C})$ with $\mathbb{R}^2 \subset \mathbb{C}^2$, which is the set $X(\mathbb{R})$ from (i). Furthermore, $X(\mathbb{C})$ is biholomorphic to $\mathbb{C}/\Lambda - \{0\}$ where Λ is the lattice $\mathbb{Z} \oplus \mathbb{Z}i$ (and $\mathbb{C}/\Lambda$ means the quotient of abelian groups). This is an example of a noncompact *Riemann surface*.

(iv) Let $\mathbb{R}[\varepsilon] = \mathbb{R}[x]/(x^2)$ be the ring of dual numbers over $\mathbb{R}$. A pair of dual numbers $(a + u\varepsilon, b + v\varepsilon)$ belongs to $X(\mathbb{R}[\varepsilon])$ if and only if $(a, b) \in X(\mathbb{R})$ and (u, v) is a tangent vector to $X(\mathbb{R})$ at the point (a, b) (see Remark 1.1 below). Thus, $X(\mathbb{R}[\varepsilon])$ is naturally identified with the *tangent bundle* of the 1-dimensional manifold $X(\mathbb{R})$. As a subspace of $\mathbb{R}[\varepsilon]^2 \simeq \mathbb{R}^4$, $X(\mathbb{R}[\varepsilon])$ is diffeomorphic to $(S^1 \times \mathbb{R}) \sqcup (\mathbb{R} \times \mathbb{R})$.

(v) We can consider solutions in finite fields:



(vi) In analogy with (iv), we can interpret $X(R[\varepsilon])$ as the set of tangent vectors to $X(R)$ for any ring R. Given $(a,b) \in X(R)$, the set of all (u,v) such that $(a+u\varepsilon, b+v\varepsilon)$ belongs to $X(R[\varepsilon])$ is an R-submodule of R^2, called the *tangent space* of X at (a,b). The tangent spaces over the first few finite fields are as follows:



For any field k, the tangent spaces at all points of $X(k)$ are 1-dimensional k-vector spaces, except for the point $(1,0)$ in characteristic 2, where the tangent space is 2-dimensional. This reflects the fact that the point $(1,0)$ is *singular* in characteristic 2, as it is a meeting point of the two branches through $(1,0)$ and $(-1,0)$. The equation $y^2 = x^3 - x$ is said to have *bad reduction* at the prime 2, and *good reduction* at all other primes.

As the picture (iii) over the complex numbers strongly suggests, one should be able to *compactify* X by filling in the puncture. In the picture (i) over the real numbers, this corresponds to adding a point "at infinity" in the vertical direction that closes up the right-hand component. We will explain how to make sense of this compactified object in §1.3.

Remark 1.1 (Dual numbers and tangent vectors)**.** Let us give some details on the claim in (iv). Given a ring R, a polynomial $f \in R[x_1, \dots, x_n]$, and an n-tuple of dual numbers $a + u\varepsilon \in R[\varepsilon]^n$, we have

$$f(a+u\varepsilon) = f(a) + \left(\frac{\partial f}{\partial x_1}(a)u_1 + \cdots + \frac{\partial f}{\partial x_n}(a)u_n\right)\varepsilon$$

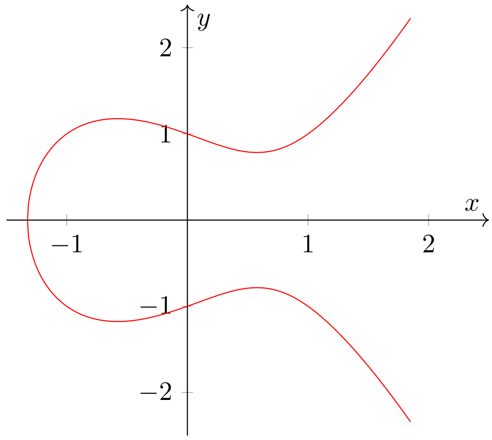
in $R[\varepsilon]$. This expression vanishes if and and only if a is a zero of f and u is orthogonal to the gradient $\nabla f(a)$. If $R = \mathbb{R}$ and if this gradient is not zero, so that $f^{-1}(0)$ is a submanifold of $\mathbb{R}^n$ in a neighborhood of a, this precisely means that u is tangent to $f^{-1}(0)$ at a. On the other hand, if this gradient is zero, then every vector is orthogonal to it. Analogous statements hold for $R = \mathbb{C}$.

Consider now the slightly different equation $y^2 = x^3 - x + 1$ and associated solution sets

$$Y(R) = \{(a,b) \in R^2 \mid b^2 = a^3 - a + 1\}.$$

Despite the similarity to the previous equation $y^2 = x^3 - x$ defining X, Y turns out to be qualitatively quite different from X:

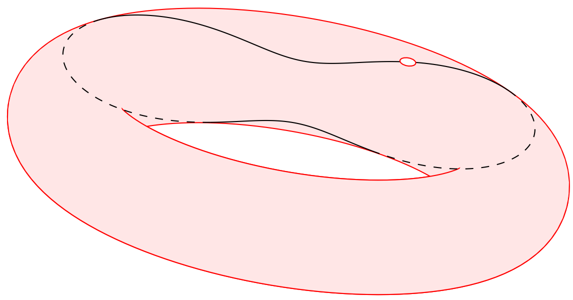
(vii) The set $Y(\mathbb{R}) \subset \mathbb{R}^2$ now has a single component and is diffeomorphic to $\mathbb{R}$:



(viii) The set $Y(\mathbb{Q})$ of rational solutions is infinite, and there are exactly 12 solutions in $\mathbb{Z}$:

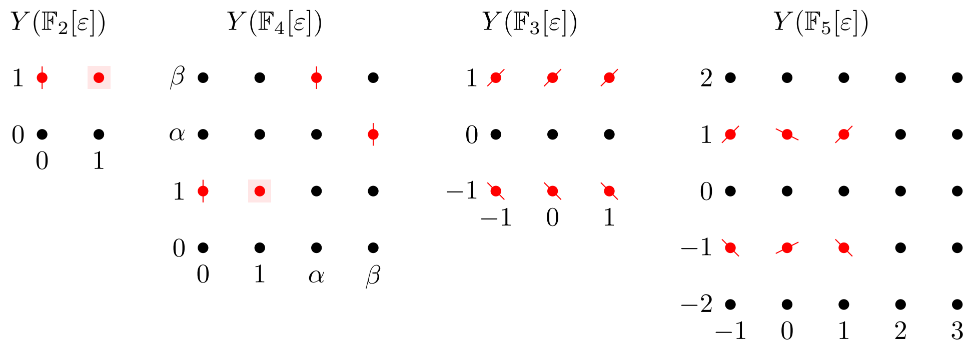
$$Y(\mathbb{Z}) = \{(-1, \pm 1), (0, \pm 1), (1, \pm 1), (3, \pm 5), (5, \pm 11), (56, \pm 419)\}.$$

(ix) The set $Y(\mathbb{C}) \subset \mathbb{C}^2$ is again a punctured torus, but its intersection with $\mathbb{R}^2$ now has a single component:



It is biholomorphic to $\mathbb{C}/\Lambda - \{0\}$ for some lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ where $\tau \approx \frac{1}{2} + 0.233i$. An exact expression for τ can be written using certain integrals called *elliptic integrals*, which also appear in the formula for the arc length of an ellipse; this is the origin of the term "elliptic curve". It turns out that $X(\mathbb{C})$ and $Y(\mathbb{C})$ are *not* biholomorphic, even though they are diffeomorphic.

(x) Here are the solutions over some finite fields together with their tangent spaces:

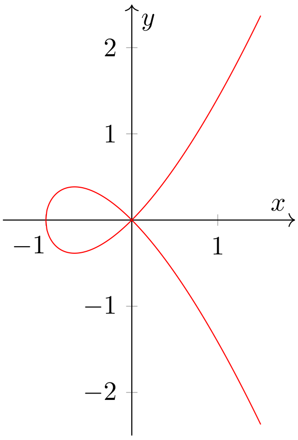


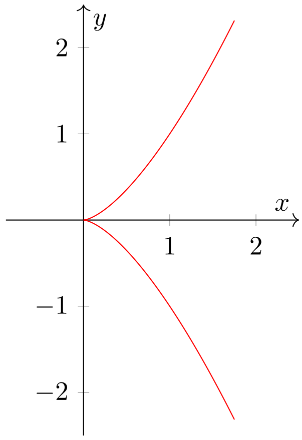
In addition to the prime 2 shown above, the equation $y^2 = x^3 - x + 1$ also has bad reduction at the prime 23, as $(0, 13)$ is a singular point of $Y(\mathbb{F}_{23})$. It has good reduction at all other primes.

1.2. **Singular cubic curves.** Consider the polynomial equations $y^2 = x^3 + x^2$ and $y^2 = x^3$. As in §1.1, they define functors $N, C\colon \mathrm{CAlg} \to \mathrm{Set}$ given by

$$N(R) = \{(a, b) \in R^2 \mid b^2 = a^3 + a^2\},$$
$$C(R) = \{(a, b) \in R^2 \mid b^2 = a^3\}.$$

We call N the *nodal cubic* and C the *cuspidal cubic*. Here is what $N(\mathbb{R})$ and $C(\mathbb{R})$ look like:





In contrast to the curves in §1.1, these curves have singularities and hence do not define submanifolds of $\mathbb{R}^2$. We can "confirm" the singular nature of the node and of the cusp by computing tangent spaces as explained in Remark 1.1. A vector (u, v) belongs to the tangent space at $(a, b) \in N(R)$ if and only if

$$2bv = (3a^2 + 2a)u.$$

For the point $(a, b) = (0, 0)$, this holds for all (u, v), so that the tangent space at $(0, 0)$ is 2-dimensional. Similarly, the equation for the tangent space at $(a, b) \in C(R)$ is

$$2bv = 3a^2u,$$

which always holds if $(a, b) = (0, 0)$. These equations also show that, for any field k, $(0, 0)$ is the only singular point in both $N(k)$ and $C(k)$.

Over the complex numbers, the nodal cubic $N(\mathbb{C}) \subset \mathbb{C}^2$ looks like a punctured *pinched torus*, i.e., a torus in which one of the circles bounding a hole is collapsed to a point; this is equivalently a punctured sphere in which two points have been identified. The cuspidal cubic $C(\mathbb{C}) \subset \mathbb{C}^2$ looks like a punctured sphere with a single thorn.

The two ways of visualizing $N(\mathbb{C})$ correspond to two ways to algebraically "resolve" the nodal singularity of N at the origin:

(i) The "pinched torus" picture suggests viewing N as a degenerate member of a *family* of curves N_λ given (for example) by the equations $y^2 = x(x+\lambda)(x+1)$. We have $N_0 = N$ and, for nonzero values of λ, $N_\lambda(\mathbb{C})$ is a punctured torus with no singular points, similar to both examples in §1.1. Thus, we can think of N as the limit of the nonsingular curves N_λ as $\lambda \to 0$. In this situation, we say that N_λ with $\lambda \neq 0$ is a *deformation* of N.

(ii) The "sphere with two points identified" picture suggests viewing N as a *quotient* of a nonsingular curve $\tilde{N}$ such that $\tilde{N}(\mathbb{C})$ is a punctured sphere. One can achieve this algebraically by a change of variable: if we replace the coordinate y by the "slope" coordinate $s = y/x$, the equation $y^2 = x^3 + x^2$ becomes $s^2 = x + 1$. If $\tilde{N}\colon \mathrm{CAlg} \to \mathrm{Set}$ is the functor defined by the latter equation, i.e.,

$$\tilde{N}(R) = \{(a, c) \in R^2 \mid c^2 = a + 1\},$$

then there is map $\tilde{N} \to N$ sending (a, c) to (a, ac), and one can check that $\tilde{N}$ has no singular points. The curve $\tilde{N}$ is called the *blowup* of N at the origin. Over the real numbers, $\tilde{N}(\mathbb{R}) \to N(\mathbb{R})$ looks like the quotient map $\mathbb{R} \twoheadrightarrow \mathbb{R}/((-1) \sim 1)$.

While an arbitrary system of polynomial equations does not in general have a nonsingular deformation as in (i), it is a deep theorem that, if we work over a field of characteristic zero, it is always possible to resolve singularities by blowing up as in (ii). Whether this is always possible in positive characteristic is a major open question in algebraic geometry.

1.3. **Compactifying affine curves.** The curves considered in §1.1 and §1.2 can be compactified by replacing the ambient 2-dimensional affine space $\mathbb{A}^2$ by the 2-dimensional projective space $\mathbb{P}^2$.

The *affine n-space* $\mathbb{A}^n$ is the functor $\mathrm{CAlg} \to \mathrm{Set}$ given by $\mathbb{A}^n(R) = R^n$. The *projective n-space* $\mathbb{P}^n$ is a functor $\mathrm{CAlg} \to \mathrm{Set}$ containing $\mathbb{A}^n$; for simplicity, we shall only define it here on a certain subcategory of CAlg (the general definition will be given in §3). Let $(\mathbb{A}^n - 0)(R) \subset R^n$ be the set of n-tuples that generate the unit ideal of R. The group of units $R^\times$ acts on the set R^n by scalar multiplication, and this action preserves the subset $(\mathbb{A}^n - 0)(R)$. If R is a local ring or a principal ideal domain, we define

$$\mathbb{P}^n(R) = (\mathbb{A}^{n+1} - 0)(R)/R^\times$$

(for arbitrary rings R, the right-hand side is only a subset of the left-hand side). We write

$$[a_0 : \dots : a_n] \in \mathbb{P}^n(R)$$

for the equivalence class of $(a_0, \dots, a_n) \in (\mathbb{A}^{n+1} - 0)(R)$. We can identify $\mathbb{A}^n(R) = R^n$ with the subset of $\mathbb{P}^n(R)$ where a_0 is a unit:

$$\mathbb{A}^n(R) \hookrightarrow \mathbb{P}^n(R), \quad (a_1, \dots, a_n) \mapsto [1 : a_1 : \dots : a_n].$$

If k is a field, then $\mathbb{P}^n(k)$ is obtained from $\mathbb{A}^n(k)$ by adding a "point at infinity" $[0 : a_1 : \dots : a_n]$ in the direction of every nonzero vector $(a_1, \dots, a_n)$; these points at infinity form a copy of $\mathbb{P}^{n-1}(k)$. Inductively, we therefore obtain a decomposition

$$\mathbb{P}^n(k) = k^n \sqcup k^{n-1} \sqcup \dots \sqcup k^0.$$

A polynomial is called *homogeneous of degree* d if it is a linear combination of monomials of degree exactly d. For example, $x^3 + x^2y - 2xz^2 \in \mathbb{Z}[x, y, z]$ is homogeneous of degree 3. If $f \in R[x_0, \dots, x_n]$ is homogeneous of degree d and $(a_0, \dots, a_n) \in R^{n+1}$, then

$$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$$

for any $\lambda \in R^\times$. Because of this, the statement "f vanishes at $(a_0, \dots, a_n)$" depends only on the point $[a_0 : \dots : a_n]$ in $\mathbb{P}^n(R)$. In other words, a *homogeneous* polynomial equation in $n + 1$ variables has a well-defined solution set in projective n-space.

Let us now return to the equation $y^2 = x^3 - x$ from §1.1 defining the subfunctor $X \subset \mathbb{A}^2$. We can *homogenize* this equation by introducing a new variable w and multiplying each term by the minimal power of w so that the equation becomes homogeneous: this yields the degree 3 homogeneous equation $wy^2 = x^3 - w^2x$. Let $\bar{X} \subset \mathbb{P}^2$ be the subfunctor of solutions to this equation, given by

$$\bar{X}(R) = \{[s : a : b] \in \mathbb{P}^2(R) \mid sb^2 = a^3 - s^2a\}$$

(this makes sense for an arbitrary ring R, although we have only defined it so far when R is a local ring or a principal ideal domain). If we set $s = 1$, we recover precisely the subset $X(R)$ of $\mathbb{A}^2(R) \subset \mathbb{P}^2(R)$:

$$X = \bar{X} \cap \mathbb{A}^2.$$

On the other hand, we can find the solutions "at infinity" by setting $s = 0$, which yields the equation $a^3 = 0$. If k is a field, we see that the only point of $\bar{X}(k)$ with $s = 0$ is $[0 : 0 : 1]$, which is the point at infinity in the vertical direction $(0, 1)$:

$$\bar{X}(k) = X(k) \sqcup \{[0 : 0 : 1]\} \subset \mathbb{P}^2(k).$$

As we surmised in §1.1, $\bar{X}(\mathbb{R})$ is a submanifold of the real projective plane $\mathbb{P}^2(\mathbb{R})$ diffeomorphic to $S^1 \sqcup S^1$, while $\bar{X}(\mathbb{C})$ is a complex submanifold of the complex projective plane $\mathbb{P}^2(\mathbb{C})$, which is diffeomorphic to a torus $S^1 \times S^1$ and biholomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$.

One can still define tangent spaces via dual numbers. Let us work out the tangent space at the point at infinity $[0 : 0 : 1] \in \bar{X}(R)$ for a local ring R. Note that $R[\varepsilon]$ is again local and $R[\varepsilon]^\times = R^\times + R\varepsilon$. Given $[s : a : b] \in \bar{X}(R)$, a point $[s + t\varepsilon : a + u\varepsilon : b + v\varepsilon]$ belongs to $\bar{X}(R[\varepsilon])$ if and only if

$$(b^2 + 2sa)t + (-3a^2 + s^2)u + 2sbv = 0.$$

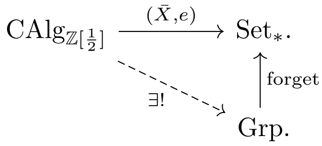
If $[s : a : b] = [0 : 0 : 1]$, this reduces to $t = 0$, so that tangent vectors at infinity have the form $[0 : u\varepsilon : 1 + v\varepsilon] = [0 : u\varepsilon : 1]$. Thus, we see that $u \mapsto [0 : u\varepsilon : 1]$ is a bijection from R to the tangent space at $[0 : 0 : 1]$. Since this is a free R-module of rank 1, the point $[0 : 0 : 1]$ is nonsingular.

Similarly, the affine curve $Y \subset \mathbb{A}^2$ defined by the equation $y^2 = x^3 - x + 1$ is compactified to the projective curve $\bar{Y} \subset \mathbb{P}^2$ given by

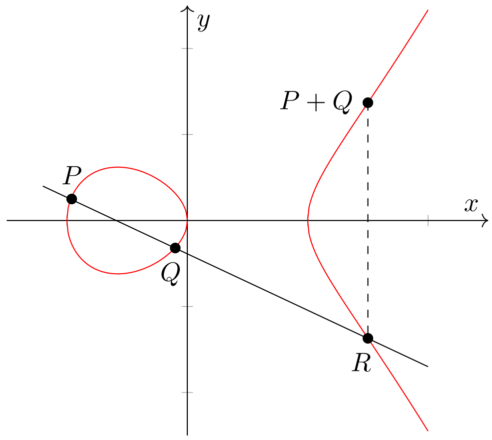
$$\bar{Y}(R) = \{[s : a : b] \in \mathbb{P}^2(R) \mid sb^2 = a^3 - s^2a + s^3\}.$$

For a field k, we have $\bar{Y}(k) = Y(k) \sqcup \{[0 : 0 : 1]\}$ as before. The real solution set $\bar{Y}(\mathbb{R})$ is now diffeomorphic to a circle, while $\bar{Y}(\mathbb{C})$ is diffeomorphic to a torus and biholomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. As in the affine case, $\bar{X}(\mathbb{C})$ and $\bar{Y}(\mathbb{C})$ are diffeomorphic but not biholomorphic. Finally, the nodal cubic N and cuspidal cubic C from §1.2 also have compactifications $\bar{N}, \bar{C} \subset \mathbb{P}^2$, which over a field add the single point at infinity $[0 : 0 : 1]$. Over the complex numbers, this has the effect of filling in the puncture.

Away from the primes of bad reduction (which means: when restricted to rings in which these primes are invertible), $\bar{X}$ and $\bar{Y}$ are examples of *elliptic curves*. A remarkable fact is that elliptic curves have an essentially unique group structure. More precisely, if we choose a point e in $\bar{X}(\mathbb{Z}[\frac{1}{2}])$ to serve as the unit element, there is a unique lift



Moreover, this lift lands in the subcategory $\mathrm{Ab} \subset \mathrm{Grp}$ of abelian groups. Over the complex numbers, the group structure on $\bar{X}(\mathbb{C})$ is that of the quotient $\mathbb{C}/\Lambda$ (with unit 0). In general, if we take the unit to be the point at infinity, then the group law is determined by the requirement that $P + Q + R = 0$ whenever P, Q, and R lie on a line (if $P = Q$, this means that R lies on the tangent line at P, and if P is the point at infinity, this means that Q and R lie on the same vertical line). Here is an illustration over the real numbers:



The same discussion applies to the elliptic curve $\bar{Y}\colon \mathrm{CAlg}_{\mathbb{Z}[\frac{1}{46}]} \to \mathrm{Set}$ (recall that Y has bad reduction at the primes 2 and 23, and we have to invert both to get an elliptic curve).

1.4. **Fermat curves.** Let $n \in \mathbb{N}$ and consider the homogeneous equation $x^n = y^n + z^n$. Let $X_n(R)$ be its set of solutions in the projective plane $\mathbb{P}^2(R)$ (which we have only defined so far when R is a local ring or principal ideal domain):

$$X_n(R) = \{[a : b : c] \in \mathbb{P}^2(R) \mid a^n = b^n + c^n\}.$$

(i) For $n = 0$, the equation is $1 = 1 + 1$, which holds in a ring R if and only if R is the zero ring. In other words, $X_0(R) = \varnothing$ if $R \neq 0$ and $X_0(0) = \{0\}$. The functor X_0 is the so-called *empty scheme*.

(ii) For $n = 1$, the equation $x = y + z$ cuts out a line in $\mathbb{P}^2$, which is isomorphic to $\mathbb{P}^1$. Indeed, there is a natural bijection

$$\mathbb{P}^1 \xrightarrow{\sim} X_1, \quad [a : b] \mapsto [a + b : a : b].$$

The real projective line $\mathbb{P}^1(\mathbb{R})$ is diffeomorphic to a circle, while the complex projective line $\mathbb{P}^1(\mathbb{C})$, also called the *Riemann sphere*, is diffeomorphic to a sphere.

(iii) For $n = 2$, we have the equation $x^2 = y^2 + z^2$. The solutions to this equation in $\mathbb{Z}^3$ are the so-called *Pythagorean triples*. It turns out that these can be explicitly determined, as we now explain. Call a Pythagorean triple (a, b, c) *primitive* if $a > 0$ and $(a, b, c) = \mathbb{Z}$. Every Pythagorean triple has the form $n(a, b, c)$ for an integer n and a primitive Pythagorean triple (a, b, c) (which are uniquely determined if $n \neq 0$). Sending (a, b, c) to $[a : b : c]$ defines a bijection between the set of primitive Pythagorean triples and $X_2(\mathbb{Z})$. Furthermore, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ induces a bijection $X_2(\mathbb{Z}) \xrightarrow{\sim} X_2(\mathbb{Q})$, with inverse given by clearing denominators (this is true for X_n for all n and reflects the fact that the scheme X_n is *proper*).

Now, for any local $\mathbb{Z}[\frac{1}{2}]$-algebra R (in fact any $\mathbb{Z}[\frac{1}{2}]$-algebra), the map

$$\sigma\colon \mathbb{P}^1(R) \to X_2(R), \quad [a : b] \mapsto [a^2 + b^2 : a^2 - b^2 : 2ab]$$

is well-defined and bijective. The inverse is given by

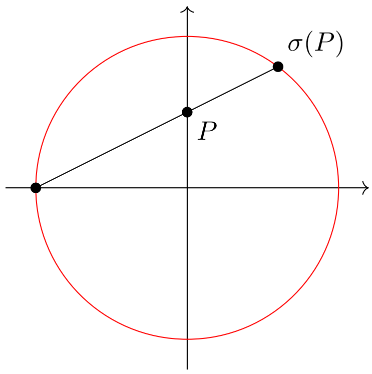
$$[u : v : w] \mapsto \begin{cases} [u + v : w] & \text{if } u + v \in R^\times, \\ [w : u - v] & \text{if } u - v \in R^\times. \end{cases}$$

In particular, for $R = \mathbb{Q}$, σ gives a complete enumeration of all primitive Pythagorean triples, which is known as *Euclid's formula*.

The map $\sigma\colon \mathbb{P}^1(k) \to X_2(k)$ has a concrete geometric interpretation when k is a subfield of $\mathbb{R}$. If $[a:b:c] \in X_2(k)$, then a cannot be zero as then b and c would also be zero (note that this is not true for $k = \mathbb{C}$). Hence,

$$X_2(k) = \{[1:b:c] \in \mathbb{P}^2(k) \mid b^2 + c^2 = 1\} = \{(b,c) \in \mathbb{A}^2(k) \mid b^2 + c^2 = 1\}.$$

In other words, $X_2(k)$ is the set of points on the unit circle in $\mathbb{R}^2$ with coordinates in k. The map σ is then the inverse stereographic projection from the point $(-1, 0)$ on the unit circle:



(iv) For $n \geq 3$, Fermat's Last Theorem states that

$$X_n(\mathbb{Q}) = X_n(\mathbb{Z}) = \begin{cases} \{[0:1:-1], [1:0:1], [1:1:0]\} & \text{if } n \text{ is odd,} \\ \{[1:0:\pm 1], [1:\pm 1:0]\} & \text{if } n \text{ is even.} \end{cases}$$

Away from the prime 3, $X_3\colon \mathrm{CAlg}_{\mathbb{Z}[\frac{1}{3}]} \to \mathrm{Set}$ is another example of an elliptic curve, which has a unique group structure with unit element $[0:1:-1]$.

1.5. **Affine, projective, and general schemes.** In §1.1 and §1.2 we saw examples of *affine schemes*, and in §1.3 and §1.4 we saw examples of *projective schemes*. The former are the solutions in affine space of systems of polynomial equations, while the latter are the solutions in projective space of systems of homogeneous polynomial equations. If we allow *inequations* in addition to equations, we obtain the notions of *quasi-affine* and *quasi-projective* schemes. We will study affine schemes in §2 and projective schemes in §3. All of these objects are examples of *schemes*, which we will finally define in §7. The following diagram summarizes the situation:

$$\begin{array}{ccccc} \{\text{affine schemes}\} & \subset & \{\text{quasi-affine schemes}\} & & \\ & & \cap & & \\ \{\text{projective schemes}\} & \subset & \{\text{quasi-projective schemes}\} & \subset & \{\text{schemes}\}. \end{array}$$

(Strictly speaking, quasi-projective schemes are defined using polynomials in finitely many variables; for the vertical inclusion to hold, one should either remove this finiteness condition on the quasi-projective side or add it on the quasi-affine side.)

(i) An example of a quasi-affine scheme that is not affine is the *punctured affine n-space* $\mathbb{A}^n - 0$ for $n \geq 2$, which is defined as

$$\mathbb{A}^n - 0\colon \mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto \{a \in R^n \mid (a) = R\}.$$

As the notation suggets, this is in a precise sense the complement of 0 in $\mathbb{A}^n$, where 0 is the joint vanishing locus of the coordinate functions $x_1, \dots, x_n$ on $\mathbb{A}^n$.

(ii) An example of a quasi-projective scheme that is neither projective nor quasi-affine is the *punctured projective n-space* $\mathbb{P}^n - 0$ for $n \geq 2$. This is again the complement of 0 in $\mathbb{P}^n$, where 0 is the vanishing locus of the projective coordinates $x_1, \dots, x_n$ on $\mathbb{P}^n$. For a local ring R, we have

$$(\mathbb{P}^n - 0)(R) = \{[a_0 : \dots : a_n] \in \mathbb{P}^n(R) \mid (a_1, \dots, a_n) = R\}.$$

(iii) Schemes that are not quasi-projective are more difficult to come by, but they can be constructed by explicitly gluing affine schemes. The simplest example is the *affine line with doubled origin*, which is the functor

$$\mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto \{(f, e) \mid f \in R,\ e \in R/(f), \text{ and } e^2 = e\}.$$

2. Affine geometry

2.1. **Affine spaces.** Affine geometry studies the solutions in affine spaces to systems of polynomial equations, while projective geometry studies the solutions in projective spaces to systems of homogeneous polynomial equations. In both cases, the solutions form a functor $\mathrm{CAlg}_k \to \mathrm{Set}$ from the category of k-algebras to the category of sets. Such functors are the basic objects of algebraic geometry:

Definition 2.1 (Algebraic functor)**.** Let k be a ring. An *algebraic k-functor* is a functor $\mathrm{CAlg}_k \to \mathrm{Set}$. An algebraic $\mathbb{Z}$-functor is simply called an *algebraic functor*. Given an algebraic k-functor X and a k-algebra R, the elements of $X(R)$ are called the *R-valued points* or *R-points* of X.

A basic example is given by affine spaces:

Definition 2.2 (Affine space)**.** Let I be a set. The *affine I-space* over k is the algebraic k-functor

$$\mathbb{A}^I_k\colon \mathrm{CAlg}_k \to \mathrm{Set}, \quad R \mapsto R^I.$$

We simply write $\mathbb{A}^I$ when $k = \mathbb{Z}$. For $n \geq 0$, the *affine n-space* over k is $\mathbb{A}^n_k = \mathbb{A}^{\{1,\dots,n\}}_k$. It is also called the *affine line* if $n = 1$ and the *affine plane* if $n = 2$.

Remark 2.3.

(i) $\mathbb{A}^0_k$ is a final object $*$ of $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$.
(ii) $\mathbb{A}^1_k$ is isomorphic to the forgetful functor $\mathrm{CAlg}_k \to \mathrm{Set}$.
(iii) $\mathbb{A}^I_k$ is contravariantly functorial in the set I: a map $f\colon J \to I$ induces a natural transformation $\mathbb{A}^I_k \to \mathbb{A}^J_k$ given by precomposition with f.
(iv) By the universal property of polynomial rings, the functor $\mathbb{A}^I_k$ is represented by the polynomial k-algebra $k[x_i \mid i \in I]$, i.e., there is an isomorphism

$$\mathbb{A}^I_k \simeq \mathrm{Map}(k[x_i \mid i \in I], -)\colon \mathrm{CAlg}_k \to \mathrm{Set}.$$

Indeed, given an I-tuple $(r_i)_{i \in I} \in R^I$, there is a unique k-algebra map $k[x_i \mid i \in I] \to R$ sending x_i to r_i.

2.2. **Presheaves.** We start with some categorical preliminaries on set-valued functors, also known as *presheaves*.

Definition 2.4 (Presheaves)**.** Let $\mathcal{C}$ be a category. A *presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. We denote by

$$\mathrm{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

the category of presheaves on $\mathcal{C}$. More generally, given an arbitrary category $\mathcal{E}$, an *$\mathcal{E}$-valued presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{E}$.

For example, an algebraic k-functor is exactly a presheaf on $\mathrm{CAlg}_k^{\mathrm{op}}$.

Remark 2.5. The category $\mathrm{P}(\mathcal{C})$ always admits limits and colimits, which are computed "pointwise" in the category of sets. Many properties of the category of sets are thereby inherited by the category of presheaves $\mathrm{P}(\mathcal{C})$, such as the fact that filtered colimits commute with finite limits, the fact the monomorphisms and epimorphisms are effective, etc.

Definition 2.6 (Yoneda embedding)**.** Let $\mathcal{C}$ be a category. The *Yoneda embedding* of $\mathcal{C}$ is the functor

$$\text{よ}\colon \mathcal{C} \to \mathrm{P}(\mathcal{C}), \quad \text{よ}(X) = \mathrm{Map}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}.$$

A presheaf on $\mathcal{C}$ is called *representable* if it lies in the essential image of よ.

Notation 2.7 (The functor Spec)**.** When $\mathcal{C} = \mathrm{CAlg}_k^{\mathrm{op}}$, the Yoneda embedding is denoted by

$$\mathrm{Spec}\colon \mathrm{CAlg}_k^{\mathrm{op}} \to \mathrm{P}(\mathrm{CAlg}_k^{\mathrm{op}}) = \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}).$$

Thus, $\mathrm{Spec}(A)$ is the algebraic k-functor represented by the k-algebra A:

$$\mathrm{Spec}(A)(R) = \mathrm{Map}(A, R).$$

For example, $\mathbb{A}^I_k \simeq \mathrm{Spec}(k[x_i \mid i \in I])$.

Definition 2.8 (Category of elements)**.** Let $\mathcal{C}$ be a category and let $F \in \mathrm{P}(\mathcal{C})$ be a presheaf on $\mathcal{C}$. The *category of elements* $\mathrm{El}(F)$ of F is defined by the cartesian square

$$\begin{array}{ccc} \mathrm{El}(F) & \longrightarrow & (\mathrm{Set}_*)^{\mathrm{op}} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{forget}} \\ \mathcal{C} & \xrightarrow{F} & \mathrm{Set}^{\mathrm{op}}. \end{array}$$

It is also denoted by $\int F$. Explicitly, objects of $\mathrm{El}(F)$ are pairs (X,x) with $X \in \mathcal{C}$ and $x \in F(X)$, and morphisms $(X,x) \to (Y,y)$ are morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that $f^*(y) = x$.

Theorem 2.9 (Properties of the Yoneda embedding)**.** *Let $\mathcal{C}$ be a category.*

(i) (The Yoneda Lemma) *Let $X \in \mathcal{C}$ and $F \in \mathrm{P}(\mathcal{C})$. Then the map*

$$\mathrm{Map}(\text{よ}(X), F) \to F(X), \quad f \mapsto f(\mathrm{id}_X),$$

is a bijection with inverse $x \mapsto ((f\colon Y \to X) \mapsto f^*(x))$.

(ii) *The Yoneda embedding $\text{よ}\colon \mathcal{C} \to \mathrm{P}(\mathcal{C})$ is fully faithful.*

(iii) *The Yoneda embedding $\text{よ}\colon \mathcal{C} \to \mathrm{P}(\mathcal{C})$ preserves all limits that exist in $\mathcal{C}$.*

(iv) *Every presheaf $F \in \mathrm{P}(\mathcal{C})$ is canonically a colimit of representable presheaves:*

$$\operatorname{colim}\left(\mathrm{El}(F) \xrightarrow{\text{forget}} \mathcal{C} \xrightarrow{\text{よ}} \mathrm{P}(\mathcal{C})\right) \xrightarrow{\simeq} F.$$

For the next two statements, assume that $\mathcal{C}$ is small.

(v) (Universal property of よ) *Let $\mathcal{E}$ be a cocomplete category. Then the functor*

$$\text{よ}^*\colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{P}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E}),$$

is an equivalence of categories, where $\mathrm{Fun}^{\mathrm{colim}}$ denotes the category of colimit-preserving functors; the inverse is given by left Kan extension along よ. In particular, any functor $\mathcal{C} \to \mathcal{E}$ extends uniquely (up to unique isomorphism) to a colimit-preserving functor $\mathrm{P}(\mathcal{C}) \to \mathcal{E}$.

(vi) *If $\mathcal{E}$ is any category, then any colimit-preserving functor $K\colon \mathrm{P}(\mathcal{C}) \to \mathcal{E}$ has a right adjoint $\mathcal{E} \to \mathrm{P}(\mathcal{C})$ given by $e \mapsto \mathrm{Map}(K(\text{よ}(-)), e)$.*

Remark 2.10.

(i) By the Yoneda Lemma, the category of elements of a presheaf $F \in \mathrm{P}(\mathcal{C})$ can equivalently be described as the pullback

$$\begin{array}{ccc} \mathrm{El}(F) & \longrightarrow & \mathrm{P}(\mathcal{C})_{/F} \\ \big\downarrow & & \big\downarrow{\scriptstyle \text{forget}} \\ \mathcal{C} & \xrightarrow{\text{よ}} & \mathrm{P}(\mathcal{C}), \end{array}$$

whose objects are pairs (X,x) with $X \in \mathcal{C}$ and $x\colon \text{よ}(X) \to F$. Theorem 2.9(iv) then says that every presheaf is the colimit of all representable presheaves mapping to it.

(ii) By the full faithfulness of the Yoneda embedding, we have $\mathrm{El}(\text{よ}(X)) \simeq \mathcal{C}_{/X}$. Together with Theorem 2.9(iv), this shows that a presheaf is representable if and only if its category of elements has a final object (which is then the representing object).

Corollary 2.11. *Let $F, G \in \mathrm{P}(\mathcal{C})$ be presheaves. Then*

$$\mathrm{Map}(F,G) = \lim_{(X,x)\in \mathrm{El}(F)} G(X).$$

Example 2.12. By Theorem 2.9(iii), the functor

$$\mathrm{Spec}\colon \mathrm{CAlg}_k^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$$

preserves all limits. Moreover, limits in $\mathrm{CAlg}_k^{\mathrm{op}}$ are colimits in CAlg_k. For example, k is the initial object of CAlg_k, so that $\mathrm{Spec}(k)$ is the final object of $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$. The coproduct of two k-algebras A and B is the tensor product $A \otimes_k B$, so that

$$\mathrm{Spec}(A \otimes_k B) \simeq \mathrm{Spec}(A) \times \mathrm{Spec}(B)$$

in $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$. More generally, the pushout of a diagram $A \leftarrow C \to B$ in CAlg_k is the relative tensor product $A \otimes_C B$, so that

$$\mathrm{Spec}(A \otimes_C B) \simeq \mathrm{Spec}(A) \times_{\mathrm{Spec}(C)} \mathrm{Spec}(B).$$

Remark 2.13 (Algebraic structures on presheaves)**.** Algebraic objects like monoids, groups, abelian groups, rings, modules over a ring, etc., make sense in any category with finite products. In categories of presheaves, since finite products are computed objectwise, algebraic objects are the same as presheaves valued in the category of algebraic objects of the same type in Set. For example, abelian group objects in presheaves are the same as presheaves of abelian groups:

$$\mathrm{Ab}(\mathrm{P}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}).$$

Thus, given a presheaf $F \in \mathrm{P}(\mathcal{C})$, equipping F with an abelian group structure is equivalent to lifting F along the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$:

$$\begin{array}{ccc} & & \mathrm{Ab} \\ & \nearrow & \downarrow{\scriptstyle \mathrm{forget}} \\ \mathcal{C}^{\mathrm{op}} & \xrightarrow[F]{} & \mathrm{Set}. \end{array}$$

2.3. **Polynomial equations.**

Definition 2.14 (System of polynomial equations)**.** Let k be a ring and let I and J be sets. A *system of J polynomial equations in I variables* over k is a J-tuple $\Sigma = (f_j)_{j\in J}$ in the polynomial ring $k[x_i \,|\, i \in I]$. We denote by (Σ) the ideal in $k[x_i \,|\, i \in I]$ generated by $(f_j)_{j\in J}$ and by $k[\Sigma]$ the k-algebra $k[x_i \,|\, i \in I]/(\Sigma)$.

Remark 2.15. Every k-algebra R is isomorphic to $k[\Sigma]$ for some system of polynomial equations Σ. A choice of isomorphism $R \simeq k[\Sigma]$ is exactly a *presentation* of R by generators and relations.

Given a system of polynomial equations over k, we can consider its solutions in any k-algebra. To that end, recall that there is, for any k-algebra R, an *evaluation map*

$$k[x_i \mid i \in I] \times R^I \to R, \quad (f, a) \mapsto f(a),$$

which is defined as follows: for each $a \in R^I$, $f \mapsto f(a)$ is the unique k-algebra map $k[x_i \mid i \in I] \to R$ sending x_i to a_i.

Definition 2.16 (Vanishing locus)**.** Let $F \subset k[x_i \,|\, i \in I]$ be a subset. The *vanishing locus* of F in $\mathbb{A}^I_k$ is the subfunctor $\mathrm{V}(F) \subset \mathbb{A}^I_k$ given by

$$\mathrm{V}(F)(R) = \{a \in R^I \mid f(a) = 0 \text{ for all } f \in F\} \subset R^I.$$

This is indeed a subfunctor: for any k-algebra map $R \to S$, the induced map $R^I \to S^I$ sends $\mathrm{V}(F)(R)$ to $\mathrm{V}(F)(S)$.

Remark 2.17. It is clear that the vanishing locus of F depends only on the ideal generated by F: if $(F) = (F')$, then $\mathrm{V}(F) = \mathrm{V}(F')$. We will see below that the converse also holds (Corollary 2.25).

Definition 2.18 (Solution functor)**.** Let $\Sigma = (f_j)_{j\in J}$ be a system of J polynomial equations in I variables over k. Its *solution functor* $\mathrm{Sol}_\Sigma \colon \mathrm{CAlg}_k \to \mathrm{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbb{A}^I_k$:

$$\mathrm{Sol}_\Sigma = \mathrm{V}(\{f_j \mid j \in J\}) \subset \mathbb{A}^I_k.$$

By the universal property of polynomial rings, there is a one-to-one correspondence between systems of J polynomial equations in I variables and k-algebra maps

$$k[x_j \mid j \in J] \to k[x_i \mid i \in I].$$

By the Yoneda lemma, these are in turn equivalent to natural transformations

$$\mathbb{A}^I_k \to \mathbb{A}^J_k \colon \mathrm{CAlg}_k \to \mathrm{Set}.$$

Unraveling these equivalences, the map $\mathbb{A}^I_k \to \mathbb{A}^J_k$ corresponding to a system $\Sigma = (f_j)_{j\in J}$ is given on a k-algebra R by

$$R^I \to R^J, \quad a \mapsto (f_j(a))_{j\in J}.$$

By definition, the solution functor Sol_Σ is the kernel of this map, i.e., there is a pullback square

$$\begin{array}{ccc} \mathrm{Sol}_\Sigma & \hookrightarrow & \mathbb{A}^I_k \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & \mathbb{A}^J_k, \end{array} \tag{2.19}$$

where 0 is the subfunctor of $\mathbb{A}^J_k$ given by $0(R) = \{0\} \subset R^J$.

Definition 2.20 (Affine scheme)**.** A functor $\mathrm{CAlg}_k \to \mathrm{Set}$ is called an *affine k-scheme* if it is isomorphic to Sol_Σ for some system of polynomial equations Σ over k. We denote by $\mathrm{Aff}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ the full subcategory spanned by the affine k-schemes. An *affine scheme* is an affine $\mathbb{Z}$-scheme.

Example 2.21. The affine I-space $\mathbb{A}^I_k$ is an affine k-scheme, as it is the solution functor of the empty system of equations in I variables.

Lemma 2.22. *Let* Σ *be a system of polynomial equations over* k*. Then the solution functor* Sol_Σ *is represented by the* k*-algebra* $k[\Sigma]$*, i.e., there is an isomorphism*

$$\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma])\colon \mathrm{CAlg}_k \to \mathrm{Set}.$$

Theorem 2.23 (Characterization of affine schemes)**.** *Let* k *be a ring. The following conditions are equivalent for an algebraic* k*-functor* $X\colon \mathrm{CAlg}_k \to \mathrm{Set}$*:*

(i) X *is an affine* k*-scheme.*
(ii) X *is representable, i.e., isomorphic to* $\mathrm{Spec}(A)$ *for some* k*-algebra* A*.*
(iii) X *preserves limits and is accessible*[1]*.*

Corollary 2.24. *The Yoneda embedding of* $\mathrm{CAlg}_k^{\mathrm{op}}$ *induces an equivalence of categories*

$$\mathrm{Spec}\colon \mathrm{CAlg}_k^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Aff}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}).$$

Under this equivalence, the affine k*-scheme* Sol_Σ *corresponds to the* k*-algebra* $k[\Sigma]$*.*

Under the equivalence of Corollary 2.24, the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbb{A}^I_k$ of affine k-schemes corresponds to the quotient map $k[x_i \mid i \in I] \twoheadrightarrow k[\Sigma]$. This implies the following result:

Corollary 2.25 (Functorial Nullstellensatz)**.** *Sending a subset* $F \subset k[x_i \mid i \in I]$ *to its vanishing locus* $\mathrm{V}(F) \subset \mathbb{A}^I_k$ *induces an order-reversing bijection*

$$\mathrm{V}\colon \{\text{ideals in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\text{vanishing loci in } \mathbb{A}^I_k\}.$$

Example 2.26. Consider the following systems of polynomial equations over $\mathbb{R}$ in one variable:

$$\Sigma_1 = (x^2+1), \quad \Sigma_2 = ((x^2+1)^2), \quad \Sigma_3 = (x^2+x+1), \quad \Sigma_4 = (x^4+1).$$

Then:

$$\begin{array}{llll} \mathrm{Sol}_{\Sigma_1}(\mathbb{R}) = \varnothing & \mathrm{Sol}_{\Sigma_2}(\mathbb{R}) = \varnothing & \mathrm{Sol}_{\Sigma_3}(\mathbb{R}) = \varnothing & \mathrm{Sol}_{\Sigma_4}(\mathbb{R}) = \varnothing, \\ \mathrm{Sol}_{\Sigma_1}(\mathbb{C}) = \{\pm i\} & \mathrm{Sol}_{\Sigma_2}(\mathbb{C}) = \{\pm i\} & \mathrm{Sol}_{\Sigma_3}(\mathbb{C}) = \{\zeta_3, \bar\zeta_3\} & \mathrm{Sol}_{\Sigma_4}(\mathbb{C}) = \{\pm\zeta_8, \pm\bar\zeta_8\}, \end{array}$$

where $\zeta_n = \exp\bigl(\frac{2\pi i}{n}\bigr) \in \mathbb{C}$. All four equations have the same solutions in $\mathbb{R}$. However, as the four ideals (Σ_i) in $\mathbb{R}[x]$ are pairwise distinct, they define four different subfunctors of $\mathbb{A}^1_{\mathbb{R}}$ by Corollary 2.25. The solutions in $\mathbb{C}$ distinguish them, except for Sol_{Σ_1} and Sol_{Σ_2}. To see that $\mathrm{Sol}_{\Sigma_1} \neq \mathrm{Sol}_{\Sigma_2}$ as subfunctors of $\mathbb{A}^1_{\mathbb{R}}$, we can compute the solutions in the $\mathbb{R}$-algebra $\mathbb{C}[\varepsilon]$ of dual complex numbers (where $\varepsilon^2 = 0$):

$$\mathrm{Sol}_{\Sigma_1}(\mathbb{C}[\varepsilon]) = \{\pm i\}, \quad \mathrm{Sol}_{\Sigma_2}(\mathbb{C}[\varepsilon]) = \{\pm i + a\varepsilon \mid a \in \mathbb{C}\}.$$

On the other hand, the associated $\mathbb{R}$-algebras are

$$\mathbb{R}[\Sigma_1] \simeq \mathbb{C}, \quad \mathbb{R}[\Sigma_2] \simeq \mathbb{C}[\varepsilon], \quad \mathbb{R}[\Sigma_3] \simeq \mathbb{C}, \quad \mathbb{R}[\Sigma_4] \simeq \mathbb{C} \times \mathbb{C}.$$

By Lemma 2.22, Sol_{Σ_1} and Sol_{Σ_3} are both isomorphic to $\mathrm{Spec}(\mathbb{C})$. The different ideals (Σ_1) and (Σ_3) correspond to two different embeddings of the affine $\mathbb{R}$-scheme $\mathrm{Spec}(\mathbb{C})$ into $\mathbb{A}^1_{\mathbb{R}}$, and the systems Σ_1 and Σ_3 themselves are two different presentations of the $\mathbb{R}$-algebra $\mathbb{C}$.

Remark 2.27. In summary, given a system of polynomial equations Σ over k, we have the following relations between Σ and Sol_Σ:

(i) The data of the pullback square (2.19) is equivalent to the data of Σ itself.
(ii) The data of the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbb{A}^I_k$ is equivalent to the data of the ideal (Σ) in the polynomial ring $k[x_i \mid i \in I]$.
(iii) The data of the affine k-scheme Sol_Σ alone is equivalent to the data of the k-algebra $k[\Sigma]$.

This can be compared with the following types of data in differential geometry:

(i) A smooth manifold M given as the vanishing locus of a smooth function $\mathbb{R}^n \to \mathbb{R}^m$.
(ii) A smooth manifold M given as a closed submanifold of $\mathbb{R}^n$.
(iii) A smooth manifold M.

Smooth manifolds are the basic objects of interest in differential geometry. Embedding a manifold M into a Euclidean space or realizing it as the vanishing locus of a function are often useful ways to understand M, but we do not consider this additional data to be part of the manifold M itself. The situation in algebraic geometry is entirely similar: the basic objects of interest are affine schemes. Any affine scheme X can be embedded into an affine space ($X \hookrightarrow \mathbb{A}^I$) or realized as the solution functor of a system of polynomial equations ($X \simeq \mathrm{Sol}_\Sigma$), but this data is not part of the affine scheme X itself.

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[1]Accessibility of X is technical condition saying that X is a *small* colimit of representables. It is equivalent to the condition that X preserves κ-filtered colimits for some infinite cardinal κ, which is usually easy to check in practice.

A key difference between differential geometry and algebraic geometry is that it is much easier to embed smooth manifolds into $\mathbb{R}^n$ than it is to embed schemes into $\mathbb{A}^n$. In fact, the former is always possible under mild technical assumptions[2] (which are usually taken as part of the definition of smooth manifold), but many interesting schemes are not affine. For example, the real projective space $\mathbb{P}^n(\mathbb{R})$ can be embedded in the Euclidean space $\mathbb{R}^{2n}$, but we will see that the algebraic projective space $\mathbb{P}^n$ with $n \geq 1$ cannot be embedded in $\mathbb{A}^N$ for any N.

Remark 2.28 (Systems of linear equations)**.** Let us spell out the analogy with linear algebra. A system Λ of J linear equations in I variables over a ring k is a J-indexed family in the free k-module $k^{(I)}$, or equivalently a k-linear map $k^{(J)} \to k^{(I)}$. If $(a_{ij})_{i\in I, j\in J}$ is the corresponding $I \times J$-matrix, a solution to Λ in a k-module M is a family $(m_i)_{i\in I}$ in M such that $\sum_{i\in I} a_{ij} m_i = 0$ for all $j \in J$. This defines a *solution functor*

$$\mathrm{Sol}_\Lambda \colon \mathrm{Mod}_k \to \mathrm{Set}.$$

Unraveling the definitions, $\mathrm{Sol}_\Lambda(M)$ is exactly the kernel of the map $M^I \to M^J$, obtained by applying $\mathrm{Map}(-, M)$ to the given map $k^{(J)} \to k^{(I)}$. It follows that $\mathrm{Sol}_\Lambda \simeq \mathrm{Map}(C, -)$, where C is the cokernel of $k^{(J)} \to k^{(I)}$. Thus, we can think of a system of linear equations over k as a k-module C equipped with a presentation, and its solution functor as the k-module C itself.

2.4. **Examples of affine schemes.**

Example 2.29 (The final scheme)**.** The constant functor $\mathrm{CAlg} \to \mathrm{Set}$ sending every ring to a one-point set is isomorphic to $\mathrm{Spec}(\mathbb{Z})$ and hence is an affine scheme. This is the final object of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$.

Example 2.30 (The empty scheme)**.** The functor

$$\mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto \begin{cases} \varnothing & \text{if } R \neq 0, \\ * & \text{if } R = 0, \end{cases}$$

is an affine scheme, isomorphic to $\mathrm{Spec}(0)$. It is called the *empty scheme* and denoted by $\varnothing$. Note that $\varnothing$ is the initial object of Aff, but it is *not* the initial object of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$, which is the constant functor with value $\varnothing$.

Example 2.31 (The idempotent classifier)**.** Let $\mathrm{Idem} \colon \mathrm{CAlg} \to \mathrm{Set}$ be the functor sending R to the set of idempotent elements of R. Then Idem is an affine scheme, isomorphic to $\mathrm{Spec}(\mathbb{Z} \times \mathbb{Z})$. Indeed, there is a bijection

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{Z} \times \mathbb{Z}, R) &\xrightarrow{\sim} \mathrm{Idem}(R), \\ \varphi &\mapsto \varphi(1, 0), \end{aligned}$$

which is natural in $R \in \mathrm{CAlg}$.

Example 2.32 (The multiplicative group)**.** The functor $\mathbb{G}_{\mathrm{m}} \colon \mathrm{CAlg} \to \mathrm{Ab}$ sending R to the group of units $R^\times$ is called the *multiplicative group*. It is an *affine group scheme*, meaning that the composition

$$\mathrm{CAlg} \xrightarrow{\mathbb{G}_{\mathrm{m}}} \mathrm{Ab} \xrightarrow{\text{forget}} \mathrm{Set}$$

is an affine scheme. Indeed, it is isomorphic to $\mathrm{Spec}(\mathbb{Z}[u^{\pm 1}])$: for every ring R, there is a bijection

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{Z}[u^{\pm 1}], R) &\xrightarrow{\sim} R^\times, \\ \varphi &\mapsto \varphi(u). \end{aligned}$$

Example 2.33 (The additive group)**.** The functor $\mathbb{G}_{\mathrm{a}} \colon \mathrm{CAlg} \to \mathrm{Ab}$ sending R to the underlying group $(R, +)$ is called the *additive group*. The composition

$$\mathrm{CAlg} \xrightarrow{\mathbb{G}_{\mathrm{a}}} \mathrm{Ab} \xrightarrow{\text{forget}} \mathrm{Set}$$

is simply the forgetful functor, also known as the affine line $\mathbb{A}^1$. Hence $\mathbb{G}_{\mathrm{a}}$ is an affine group scheme.

Example 2.34 (The matrix ring)**.** Let $n \geq 0$ and let $\mathrm{Mat}_n \colon \mathrm{CAlg} \to \mathrm{Alg}$ be the functor sending R to the associative ring of $n \times n$ matrices over R. This is an associative ring object in affine schemes. Indeed, since a matrix over R is simply a family of n^2 elements of R, the composition

$$\mathrm{CAlg} \xrightarrow{\mathrm{Mat}_n} \mathrm{Alg} \xrightarrow{\text{forget}} \mathrm{Set}$$

is isomorphic to $\mathbb{A}^{n^2}$ and hence is an affine scheme.

[2] namely: Hausdorff, second countable, and of bounded dimension

Example 2.35 (The general linear group)**.** Let $n \geq 0$ and let $\mathrm{GL}_n \colon \mathrm{CAlg} \to \mathrm{Grp}$ be the functor sending R to the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices. Then GL_n is an affine group scheme. Indeed, let $A = \mathbb{Z}[x_{ij} \mid (i,j) \in \{1, \dots, n\}^2]$ be the ring representing Mat_n, which contains the universal $n \times n$ matrix $X = (x_{ij})_{i,j}$. A matrix $M \in \mathrm{Mat}_n(R)$ is invertible if and only if its determinant $\det(M) \in R$ is a unit. Hence, for any ring R, there is an isomorphism

$$\mathrm{Map}_{\mathrm{CAlg}}(A_{\det(X)}, R) \xrightarrow{\sim} \mathrm{GL}_n(R), \qquad \varphi \mapsto (\varphi(x_{ij}))_{i,j},$$

so that $\mathrm{GL}_n \simeq \mathrm{Spec}(A_{\det(X)})$.

Example 2.36 (The special linear group)**.** Let $n \geq 0$ and let $\mathrm{SL}_n \colon \mathrm{CAlg} \to \mathrm{Grp}$ be the functor sending R to the special linear group $\mathrm{SL}_n(R)$ of $n \times n$ matrices with determinant 1. Let $X \in \mathrm{Mat}_n(A)$ be the universal $n \times n$ matrix as in Example 2.35. We then have

$$\mathrm{Map}_{\mathrm{CAlg}}(A/(\det(X) - 1), R) \xrightarrow{\sim} \mathrm{SL}_n(R), \qquad \varphi \mapsto (\varphi(x_{ij}))_{i,j},$$

so that $\mathrm{SL}_n \simeq \mathrm{Spec}(A/(\det(X) - 1))$. The subfunctor inclusions $\mathrm{SL}_n \subset \mathrm{GL}_n \subset \mathrm{Mat}_n$ correspond to the ring maps

$$A \hookrightarrow A_{\det(X)} \twoheadrightarrow A/(\det(X) - 1).$$

Example 2.37 (The affine space of a module)**.** Let k be a ring and let M be a k-module. Consider the functor $\mathbb{A}(M) \colon \mathrm{CAlg}_k \to \mathrm{Mod}_k$ defined by

$$\mathbb{A}(M)(R) = \{k\text{-linear maps } M \to R\} = (M \otimes_k R)^\vee.$$

Then $\mathbb{A}(M)$ is a k-module object in affine k-schemes. Indeed, if $\mathrm{Sym}_k(M)$ is the free k-algebra on M, there is a bijection

$$\mathrm{Map}_{\mathrm{CAlg}_k}(\mathrm{Sym}_k(M), R) \xrightarrow{\sim} \mathbb{A}(M)(R), \qquad \varphi \mapsto \varphi|M,$$

so that $\mathbb{A}(M) \simeq \mathrm{Spec}(\mathrm{Sym}_k(M))$. The affine I-space is a special case of this construction: $\mathbb{A}^I_k \simeq \mathbb{A}(k^{(I)})$.

Remark 2.38. Let k be a ring and let M be a k-module. Given Example 2.37, it is tempting to consider the following "predual" of $\mathbb{A}(M)$: define $\mathbb{A}^\vee(M) \colon \mathrm{CAlg}_k \to \mathrm{Mod}_k$ by

$$\mathbb{A}^\vee(M)(R) = M \otimes_k R.$$

There is a canonical map $\mathbb{A}^\vee(M^\vee) \to \mathbb{A}(M)$, which is an isomorphism if and only if M is a vector space (Definition 3.2). Otherwise, $\mathbb{A}^\vee(M)$ does not preserve limits and hence is not an affine k-scheme (in fact, it is not even a scheme). For that reason, the functor $\mathbb{A}^\vee(M)$ is rarely used.

2.5. **Base change.** Given a ring map $\varphi \colon k \to k'$, we can transform any system of polynomial equations Σ over k into a system $\varphi^*(\Sigma)$ over k' by applying φ to all the coefficients. More generally, many types of data over k can be transformed into data over k' using φ, a process known as *base change*, *change of coefficients*, or *extension of scalars*. Other examples are the functor $\varphi^* \colon \mathrm{Mod}_k \to \mathrm{Mod}_{k'}$ sending a k-module M to the k'-module $M \otimes_k k'$, and the functor $\varphi^* \colon \mathrm{CAlg}_k \to \mathrm{CAlg}'_k$ sending a k-algebra A to the k'-algebra $A \otimes_k k'$. Unraveling these constructions, we see that there is a canonical isomorphism of k'-algebras

$$k'[\varphi^*(\Sigma)] \simeq \varphi^*(k[\Sigma]).$$

In this section, we investigate the related process of transforming an algebraic k-functor into an algebraic k'-functor.

Theorem 2.39 (Functoriality of presheaves)**.** *Let $u \colon \mathcal{C} \to \mathcal{D}$ be a functor, and let*

$$u^* \colon \mathrm{P}(\mathcal{D}) \to \mathrm{P}(\mathcal{C}), \quad F \mapsto F \circ u,$$

be the "restriction along u" functor.

(i) *u^* admits a left adjoint $u_\sharp$ and a right adjoint u_* given by*

$$u_\sharp(F)(d) = \operatorname{colim}\Big((\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}\Big) = \operatorname*{colim}_{d \to u(c)} F(c),$$

$$u_*(F)(d) = \lim\Big((\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}\Big) = \lim_{u(c) \to d} F(c),$$

provided these colimits and limits exist (e.g., if $\mathcal{C}$ is small).

(ii) *The functor* $u_\sharp$ *extends* u*: there is a canonical isomorphism*

$$u_\sharp \circ \text{よ}_{\mathcal{C}} \simeq \text{よ}_{\mathcal{D}} \circ u.$$

(iii) *If* u *is fully faithful, then* $u_\sharp$ *and* u_* *are fully faithful.*
(iv) *If* u *is a localization, then* u^* *is fully faithful.*
(v) *If the functor* u *has a left adjoint* u_L *(resp. a right adjoint* u_R*), then there is a canonical isomorphism* $u_\sharp \simeq u_L^*$ *(resp.* $u_* \simeq u_R^*$*).*

Remark 2.40.
(i) Given $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, the presheaves $u_\sharp(F)$ and $u_*(F)$ are special cases of Kan extensions: $u_\sharp(F)$ is the left Kan extension of F along u^{op}, and $u_*(F)$ is the right Kan extension of F along u^{op}.
(ii) By the universal property of $\text{よ}_{\mathcal{C}}$, the functor $u_\sharp$ is the *unique* colimit-preserving extension of u (up to unique isomorphism). However, there is no analogous characterization of u_*.

Corollary 2.41. *Let* $\mathcal{C}$ *be a category, let* $Y \to X$ *be a morphism in* $\mathcal{C}$*, and let* $u\colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ *be the forgetful functor.*
(i) *The functor* $u^*\colon \mathrm{P}(\mathcal{C}_{/X}) \to \mathrm{P}(\mathcal{C}_{/Y})$ *has a left adjoint* $u_\sharp$ *given by*

$$u_\sharp(F)(U \to X) = \coprod_{\substack{\text{maps } U \to Y \\ \text{over } X}} F(U \to Y).$$

(ii) *If pullbacks along* $Y \to X$ *exist in* $\mathcal{C}$*, the functor* $u^*\colon \mathrm{P}(\mathcal{C}_{/X}) \to \mathrm{P}(\mathcal{C}_{/Y})$ *has a right adjoint* u_* *given by*

$$u_*(F)(U \to X) = F(U \times_X Y \to Y).$$

If k is a ring, then $\mathrm{CAlg}_k \simeq \mathrm{CAlg}_{k/}$ and hence $\mathrm{CAlg}_k^{\mathrm{op}} \simeq (\mathrm{CAlg}^{\mathrm{op}})_{/k}$. Using this identification, we obtain the following special case of Corollary 2.41 with $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}$:

Corollary 2.42. *Let* $\varphi\colon k \to k'$ *be a ring map. Then there is a triple of adjoint functors*

$$\mathrm{Fun}(\mathrm{CAlg}_{k'}, \mathrm{Set}) \underset{\varphi_*}{\overset{\varphi_\sharp}{\underset{\longleftarrow\varphi^*}{\rightrightarrows}}} \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}),$$

where:
- φ^* *is precomposition with the forgetful functor* $\mathrm{CAlg}_{k'} \to \mathrm{CAlg}_k$*, and it is the unique colimit-preserving extension of* $\varphi^*\colon \mathrm{CAlg}_k^{\mathrm{op}} \to \mathrm{CAlg}_{k'}^{\mathrm{op}}$*;*
- φ_* *is precomposition with* $\varphi^*\colon \mathrm{CAlg}_k \to \mathrm{CAlg}_{k'}$*;*
- $\varphi_\sharp$ *is given by*

$$\varphi_\sharp(X)(A) = \coprod_{\substack{k'\text{-algebra} \\ \text{structures on } A}} X(A),$$

and it is the unique colimit-preserving extension of the forgetful functor $\mathrm{CAlg}_{k'}^{\mathrm{op}} \to \mathrm{CAlg}_k^{\mathrm{op}}$.

Definition 2.43. Let $\varphi\colon k \to k'$ be a ring map, giving rise to the adjoint triple of Corollary 2.42.
(i) The functor φ^* is called *base change* or *extension of scalars* along φ and is also denoted by $X \mapsto X_{k'}$.
(ii) The functor φ_* is called *Weil restriction* or *restriction of scalars* along φ and is also denoted by R_φ or $\mathrm{R}_{k'/k}$.

Remark 2.44. Corollary 2.42 says in particular that the functors φ^* and $\varphi_\sharp$ preserve affine schemes:
(i) For a k-algebra A, $\varphi^*(\mathrm{Spec}(A)) \simeq \mathrm{Spec}(A \otimes_k k')$ in $\mathrm{Fun}(\mathrm{CAlg}_{k'}, \mathrm{Set})$. Hence, for any system of polynomial equations Σ over k, $\varphi^*(\mathrm{Sol}_\Sigma) \simeq \mathrm{Sol}_{\varphi^*(\Sigma)}$.
(ii) For a k'-algebra B, $\varphi_\sharp(\mathrm{Spec}(B)) \simeq \mathrm{Spec}(B)$ in $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$.

On the other hand, the functor φ_* does not always preserve affine schemes.

Example 2.45. Consider the affine scheme $\mathbb{G}_{\mathrm{m}}\colon \mathrm{CAlg} \to \mathrm{Set}$, $R \mapsto R^\times$. Let $\mathbb{G}_{\mathrm{m},\mathbb{C}}\colon \mathrm{CAlg}_{\mathbb{C}} \to \mathrm{Set}$ be its base change to $\mathbb{C}$, i.e., its restriction along the forgetful functor $\mathrm{CAlg}_{\mathbb{C}} \to \mathrm{CAlg}$. The Weil restriction $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})\colon \mathrm{CAlg}_{\mathbb{R}} \to \mathrm{Set}$ is given by

$$\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})(A) = \mathbb{G}_{\mathrm{m},\mathbb{C}}(A \otimes_{\mathbb{R}} \mathbb{C}) = (A \otimes_{\mathbb{R}} \mathbb{C})^\times.$$

One can check that $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$ is an affine $\mathbb{R}$-scheme: for any $\mathbb{R}$-algebra A, there is a bijection

$$\begin{aligned}\mathrm{Map}\big(\mathbb{R}[x,y,z,w]/(xz-yw-1,yz+xw),A\big) &\xrightarrow{\sim} (A\otimes_{\mathbb{R}}\mathbb{C})^\times,\\ \varphi &\mapsto \varphi(x)+i\varphi(y).\end{aligned}$$

These equations ensure that $\varphi(x)+i\varphi(y)$ is inverse to $\varphi(z)+i\varphi(w)$. More generally, if $k\subset k'$ is a finite field extension, one can show that Weil restriction $\mathrm{R}_{k'/k}$ sends affine k'-schemes to affine k-schemes.

A fundamental property of sets is that *maps of sets* are equivalent to *families of sets*: there is an equivalence of categories

$$\mathrm{Ar}(\mathrm{Set})\simeq\mathrm{Fam}(\mathrm{Set}),$$

where $\mathrm{Ar}(\mathrm{Set})=\mathrm{Fun}(\{0\to 1\},\mathrm{Set})$ is the arrow category of Set and Fam(Set) is the category whose objects are (set-indexed) families of sets $(X_i)_{i\in I}$, where a map $(X_i)_{i\in I}\to(Y_j)_{j\in J}$ consists of a map $u\colon I\to J$ and maps $X_i\to Y_{u(i)}$ for all $i\in I$. In one direction, a map $f\colon X\to I$ corresponds to the family of its fibers $(f^{-1}\{i\})_{i\in I}$. In the other direction, a family $(X_i)_{i\in I}$ corresponds to the map $\coprod_{i\in I}X_i\to I$. If we fix the target/indexing set I, we obtain an equivalence of categories

$$\mathrm{Set}_{/I}\simeq\mathrm{Set}^I.$$

The following proposition generalizes this fact to presheaves of sets (we recover the last equivalence by taking $\mathcal{C}=*$):

Proposition 2.46 (Slices of presheaf categories)**.** *Let $\mathcal{C}$ be a category and let $F\in\mathrm{P}(\mathcal{C})$ be a presheaf on $\mathcal{C}$. Then there is an equivalence of categories*

$$\mathrm{P}(\mathcal{C})_{/F}\underset{\mathrm{fib}_F}{\overset{\coprod_F}{\underset{\sim}{\leftrightarrows}}}\mathrm{P}(\mathrm{El}(F)),$$

described as follows. If $u\colon\mathrm{El}(F)\to\mathcal{C}$ is the forgetful functor, there is a tautological map $*\to u^*(F)$, whose adjoint $u_\sharp(*)\to F$ is an isomorphism.

- *Given $H\in\mathrm{P}(\mathrm{El}(F))$, the presheaf $\coprod_F H$ over F is $u_\sharp(H)\to u_\sharp(*)\simeq F$. Explicitly,*

$$\left(\coprod_F H\right)(X)=\coprod_{x\in F(X)}(H(X,x)\to *).$$

- *Given $G\in\mathrm{P}(\mathcal{C})_{/F}$, the presheaf $\mathrm{fib}_F(G)$ on $\mathrm{El}(F)$ is the pullback $u^*(G)\times_{u^*(F)}*$. Explicitly,*

$$\mathrm{fib}_F(G)(X,x)=G(X)\times_{F(X)}\{x\}\simeq\left\{\begin{array}{ccc} & & G\\ & \nearrow & \downarrow\\ よ(X) & \xrightarrow{x} & F\end{array}\right\}.$$

Specializing to the case of a representable presheaf, we get:

Corollary 2.47. *Let $\mathcal{C}$ be a category, let $X\in\mathcal{C}$, and let $u\colon\mathcal{C}_{/X}\to\mathcal{C}$ be the forgetful functor. Then the functor $u_\sharp$ induces an equivalence of categories*

$$\mathrm{P}(\mathcal{C}_{/X})\xrightarrow{\sim}\mathrm{P}(\mathcal{C})_{/よ(X)}.$$

Specializing further to $\mathcal{C}=\mathrm{CAlg}^{\mathrm{op}}$, we get the following key result:

Corollary 2.48. *Let k be a ring and let $\varphi\colon\mathbb{Z}\to k$ be the unique map. Then the functor $\varphi_\sharp$ induces an equivalence of categories*

$$\mathrm{Fun}(\mathrm{CAlg}_k,\mathrm{Set})\xrightarrow{\sim}\mathrm{Fun}(\mathrm{CAlg},\mathrm{Set})_{/\,\mathrm{Spec}(k)}.$$

Remark 2.49. Because of Corollary 2.48, algebraic geometry over a base ring k is subsumed by algebraic geometry over $\mathbb{Z}$. In other words, working in the category Fun(CAlg, Set) of algebraic functors does not restrict the generality, and we will often do so from now on. Note that the functor Spec of Notation 2.7 is independent of k, in the sense that the following square commutes:

$$\begin{array}{ccc}\mathrm{CAlg}_k^{\mathrm{op}} & \xrightarrow{\sim} & (\mathrm{CAlg}^{\mathrm{op}})_{/k}\\ {\scriptstyle\mathrm{Spec}}\downarrow & & \downarrow{\scriptstyle\mathrm{Spec}}\\ \mathrm{Fun}(\mathrm{CAlg}_k,\mathrm{Set}) & \xrightarrow{\sim} & \mathrm{Fun}(\mathrm{CAlg},\mathrm{Set})_{/\,\mathrm{Spec}(k)}.\end{array}$$

Remark 2.50. If $f\colon X' \to X$ is a morphism of algebraic functors, there is a triple of adjoint functors

$$\mathrm{Fun}(\mathrm{CAlg},\mathrm{Set})_{/X'} \begin{smallmatrix} \xrightarrow{f_\sharp} \\ \xleftarrow{f^*} \\ \xrightarrow{f_*} \end{smallmatrix} \mathrm{Fun}(\mathrm{CAlg},\mathrm{Set})_{/X},$$

where $f^*(Y) = Y \times_X X'$ and $f_\sharp(Y') = Y'$. Given Proposition 2.46, this can be seen by applying Theorem 2.39 to the functor $u\colon \mathrm{El}(X') \to \mathrm{El}(X)$. This recovers Corollary 2.42 when f is $\mathrm{Spec}(\varphi)\colon \mathrm{Spec}(k') \to \mathrm{Spec}(k)$.

2.6. **Functions.** Recall that the affine line $\mathbb{A}^1$ is the forgetful functor

$$\mathbb{A}^1\colon \mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto R.$$

Tautologically, $\mathbb{A}^1$ has a structure of ring object in $\mathrm{Fun}(\mathrm{CAlg},\mathrm{Set})$, given by the factorization

$$\begin{array}{ccc} & & \mathrm{CAlg} \\ & \overset{\mathrm{id}}{\nearrow} & \big\downarrow{\scriptstyle \mathrm{forget}} \\ \mathrm{CAlg} & \xrightarrow[\mathbb{A}^1]{} & \mathrm{Set}. \end{array}$$

Recall also that $\mathbb{G}_m$ is the subfunctor of $\mathbb{A}^1$ given by $R \mapsto R^\times$, which has a structure of abelian group (Example 2.32).

Definition 2.51 (Function and nonvanishing function)**.** Let X be an algebraic functor.

(i) A *function* on X is a map $X \to \mathbb{A}^1$. We denote by

$$\mathcal{O}(X) = \mathrm{Map}(X, \mathbb{A}^1)$$

the set of functions on X. The ring structure on $\mathbb{A}^1$ induces a ring structure on $\mathcal{O}(X)$.

(ii) An *nonvanishing function* on X is a map $X \to \mathbb{G}_\mathrm{m}$. We denote by

$$\mathcal{O}^\times(X) = \mathrm{Map}(X, \mathbb{G}_\mathrm{m})$$

the set of nonvanishing functions on X. The abelian group structure on $\mathbb{G}_\mathrm{m}$ induces an abelian group structure on $\mathcal{O}^\times(X)$.

Remark 2.52.

(i) Since $\mathbb{G}_\mathrm{m} \subset \mathbb{A}^1$, we have $\mathcal{O}^\times(X) \subset \mathcal{O}(X)$.

(ii) By the Yoneda lemma, there is a canonical isomorphism $\mathcal{O}(\mathrm{Spec}(R)) \simeq R$, i.e., the ring of functions on $\mathrm{Spec}(R)$ is R itself. Hence, when restricted to affine schemes, the functor $\mathcal{O}\colon \mathrm{Aff}^\mathrm{op} \to \mathrm{CAlg}$ is an equivalence of categories, which is inverse to Spec. Similarly, $\mathcal{O}^\times(\mathrm{Spec}(R)) \simeq R^\times$.

(iii) For a general algebraic functor X, we have

$$\mathcal{O}(X) \simeq \lim_{x\colon \mathrm{Spec}(R) \to X} R, \quad f \mapsto (f \circ x)_x,$$

where the limit is indexed by the category of elements $\mathrm{El}(X)^\mathrm{op}$ (Corollary 2.11). Since the unit group functor $R \mapsto R^\times$ preserves limits, it follows that $\mathcal{O}^\times(X)$ is precisely the unit group $\mathcal{O}(X)^\times$.

(iv) By definition, the functor

$$\mathcal{O}\colon \mathrm{Fun}(\mathrm{CAlg},\mathrm{Set})^\mathrm{op} \to \mathrm{CAlg}, \quad X \mapsto \mathcal{O}(X),$$

is limit-preserving. By (ii) and the universal property of presheaves, it is the unique limit-preserving extension of the identity $\mathrm{CAlg} \to \mathrm{CAlg}$. Similarly, $\mathcal{O}^\times$ is the unique limit-preserving extension of the functor $\mathrm{CAlg} \to \mathrm{Ab}$, $R \mapsto R^\times$.

Remark 2.53 (Size issues)**.** The statement of Remark 2.52(iv) is actually nonsensical due to "size issues". Since the category CAlg is large, the limit in the formula for $\mathcal{O}(X)$ is indexed by a large category, so that the ring $\mathcal{O}(X)$ is sometimes large. In this case, $\mathcal{O}(X)$ is not an object of CAlg, which is the category of small rings. There are two standard ways to rectify this issue:

(i) One can simply replace the target of $\mathcal{O}$ by the category $\widehat{\mathrm{CAlg}}$ of large rings. One may then also replace the category of sets in the source by the category $\widehat{\mathrm{Set}}$ of large sets. We obtain the functor

$$\mathcal{O}\colon \mathrm{Fun}(\mathrm{CAlg},\widehat{\mathrm{Set}})^\mathrm{op} \to \widehat{\mathrm{CAlg}},$$

which is the unique extension of the embedding $\mathrm{CAlg} \hookrightarrow \widehat{\mathrm{CAlg}}$ that preserves *large* limits.

(ii) One can replace the source of $\mathcal{O}$ by the subcategory $\mathrm{Fun}^{\mathrm{acc}}(\mathrm{CAlg}, \mathrm{Set})$ of *accessible* functors, which are the functors that are *small* colimits of representables. For an accessible functor X, $\mathcal{O}(X)$ is a small limit of small rings and hence is small. We therefore have a functor

$$\mathcal{O}\colon \mathrm{Fun}^{\mathrm{acc}}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{CAlg},$$

which is the unique extension of the identity $\mathrm{CAlg} \to \mathrm{CAlg}$ that preserves *small* limits. All algebraic functors that arise in practice (including all schemes) are accessible, and all relevant constructions preserve accessibility, so that this restriction does not have any undesirable consequences.

Remark 2.54 (The adjunction between Spec and $\mathcal{O}$)**.** Let X be an algebraic functor and A a ring. By Corollary 2.11 and the Yoneda lemma, we have a sequence of natural isomorphisms

$$\begin{aligned}\mathrm{Map}(X, \mathrm{Spec}(A)) &\simeq \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \mathrm{Map}(\mathrm{Spec}(R), \mathrm{Spec}(A))\\ &\simeq \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \mathrm{Map}(A, R)\\ &\simeq \mathrm{Map}\Big(A, \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} R\Big)\\ &\simeq \mathrm{Map}(A, \mathcal{O}(X)).\end{aligned}$$

Thus, modulo the size issues discussed in Remark 2.53, we have an adjunction

$$\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathrm{CAlg}^{\mathrm{op}}.$$

2.7. **Closed and open subfunctors.** Roughly speaking, a *closed* subfunctor of an algebraic functor is a subfunctor defined by the vanishing of functions, while an *open* subfunctor is one defined by the nonvanishing of functions. This terminology is borrowed from topology, where the vanishing locus of a continuous function is closed and its nonvanishing locus is open. We will see later in §5.3 that there is in fact a topological interpretation of open subfunctors, though not of closed subfunctors.

Definition 2.55 (Vanishing and nonvanishing loci)**.** Let X be an algebraic functor and $F \subset \mathcal{O}(X)$ a set of functions on X.

(i) The *vanishing locus* of F is the subfunctor $\mathrm{V}(F) \subset X$ given by

$$\mathrm{V}(F)(R) = \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in F\}.$$

(ii) The *nonvanishing locus* of F is the subfunctor of $\mathrm{D}(F) \subset X$ given by

$$\mathrm{D}(F)(R) = \{x \in X(R) \mid (f(x))_{f\in F} \text{ generates the unit ideal in } R\}.$$

Remark 2.56.

(i) It is clear that $\mathrm{V}(F)$ depends only on the ideal (F) generated by F, and $\mathrm{D}(F)$ only on the *radical* ideal $\sqrt{(F)}$ generated by F, since an ideal is the unit ideal if and only if its radical is.

(ii) We have the following implications:

$$(F) = \mathcal{O}(X) \quad\Longrightarrow\quad \mathrm{V}(F) = \varnothing_X \quad\Longleftrightarrow\quad \mathrm{D}(F) = X.$$

Here, $\varnothing_X$ is the functor with $\varnothing_X(0) = X(0)$ and $\varnothing_X(R) = \varnothing$ for $R \neq 0$. The reverse implication holds if X is affine, by Proposition 2.66 below.

Example 2.57 (Punctured affine spaces)**.** Let I be a set. The *punctured affine I-space* $\mathbb{A}^I - 0$ is the nonvanishing locus of the coordinate functions $\{x_i \mid i \in I\}$ on $\mathbb{A}^I = \mathrm{Spec}(\mathbb{Z}[x_i \mid i \in I])$. Explicitly:

$$(\mathbb{A}^I - 0)(R) = \{a \in R^I \mid (a) = R\}.$$

Note that $\mathbb{A}^1 - 0$ is another name for the subfunctor $\mathbb{G}_{\mathrm{m}} \subset \mathbb{A}^1$. An I-tuple in R generating the unit ideal is also called a *unimodular row* of length I.

Remark 2.58. A set of functions $F \subset \mathcal{O}(X)$ induces a map $f\colon X \to \mathbb{A}^F$. By definition, we have

$$\mathrm{V}(F) = f^{-1}(0) \quad\text{and}\quad \mathrm{D}(F) = f^{-1}(\mathbb{A}^F - 0).$$

Warning 2.59. The terminology suggests that $\mathrm{D}(F)$ should in some sense be the complement of $\mathrm{V}(F)$ in X. This is true when evaluated on fields, but it is not true in the category of algebraic functors. In fact, they are not even disjoint since $\mathrm{V}(F)(0) = \mathrm{D}(F)(0) = X(0)$. We will see later that $\mathrm{D}(F)$ is the complement of $\mathrm{V}(F)$ (i.e., the largest disjoint subobject) in various subcategories of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$, such as the category of schemes. Even then, the converse fails: $\mathrm{V}(F)$ cannot be the complement of $\mathrm{D}(F)$ in general, since it can happen that $\mathrm{V}(F) \neq \mathrm{V}(F')$ while $\mathrm{D}(F) = \mathrm{D}(F')$.

Proposition 2.60 (Formal properties of V and D)**.** *Let X be an algebraic functor.*

(i) *For any family $(F_i)_{i\in I}$ of subsets of $\mathcal{O}(X)$,*

$$\textstyle\bigcap_{i\in I} \mathrm{V}(F_i) = \mathrm{V}\big(\bigcup_{i\in I} F_i\big),$$
$$\textstyle\bigcup_{i\in I} \mathrm{D}(F_i) \subset \mathrm{D}\big(\bigcup_{i\in I} F_i\big),$$

and the inclusion is an equality on local rings.

(ii) *For any finite family $F_1,\dots,F_n$ of subsets of $\mathcal{O}(X)$,*

$$\mathrm{D}(F_1)\cap\dots\cap\mathrm{D}(F_n) = \mathrm{D}(F_1\dots F_n),$$
$$\mathrm{V}(F_1)\cup\dots\cup\mathrm{V}(F_n) \subset \mathrm{V}(F_1\dots F_n),$$

and the inclusion is an equality on integral domains.

Example 2.61. Since 2 and 3 generate the unit ideal in $\mathbb{Z}$, we have $\mathrm{V}(2)\cap\mathrm{V}(3) = \mathrm{V}(1) = \varnothing$ as subfunctors of $\mathrm{Spec}(\mathbb{Z})$ (where $\varnothing$ is the empty scheme of Example 2.30). On the other hand, $\mathrm{D}(2)\cup\mathrm{D}(3) \neq \mathrm{D}(1) = \mathrm{Spec}(\mathbb{Z})$. For example, $\mathrm{id}_{\mathbb{Z}} \in \mathrm{Spec}(\mathbb{Z})(\mathbb{Z})$ belongs neither to $\mathrm{D}(2)(\mathbb{Z}) = \varnothing$ nor to $\mathrm{D}(3)(\mathbb{Z}) = \varnothing$.

Proposition 2.62 (Affineness of vanishing and nonvanishing loci)**.** *Let A be a ring.*

(i) *For any subset $F\subset A$, the quotient map $A \twoheadrightarrow A/(F)$ induces an isomorphism* $\mathrm{Spec}(A/(F)) \xrightarrow{\sim} \mathrm{V}(F) \subset \mathrm{Spec}(A)$*. In particular,* $\mathrm{V}(F)$ *is affine.*

(ii) *For any $f\in A$, the localization map $A \to A_f$ induces an isomorphism* $\mathrm{Spec}(A_f) \xrightarrow{\sim} \mathrm{D}(f) \subset \mathrm{Spec}(A)$*. In particular,* $\mathrm{D}(f)$ *is affine.*

If $F\subset A$ has more than one element, $\mathrm{D}(F)\subset\mathrm{Spec}(A)$ is usually not an affine scheme. For example, $\mathbb{A}^n - 0$ is not affine for $n\geq 2$ (see Example 2.75). This motivates the following definition:

Definition 2.63 (Quasi-affine scheme)**.** Let k be a ring. An algebraic k-functor X is a *quasi-affine k-scheme* if there exists a k-algebra A and a *finite* subset $F\subset A$ such that $X\simeq \mathrm{D}(F)\subset\mathrm{Spec}(A)$. We denote by $\mathrm{QAff}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k,\mathrm{Set})$ the full subcategory of quasi-affine k-schemes. A *quasi-affine scheme* is a quasi-affine $\mathbb{Z}$-scheme.

Definition 2.64 (Closed and open subfunctors)**.** Let X be an algebraic functor.

(i) A subfunctor $Z\subset X$ is *closed* if, for every $x\colon \mathrm{Spec}(R)\to X$, $x^{-1}(Z) = \mathrm{V}(F)$ for some $F\subset R$.
(ii) A subfunctor $U\subset X$ is *open* if, for every $x\colon \mathrm{Spec}(R)\to X$, $x^{-1}(U) = \mathrm{D}(F)$ for some $F\subset R$.

Warning 2.65. Vanishing loci are always closed subfunctors and nonvanishing loci are always open subfunctors, but the converse does not hold.

The following result generalizes the functorial Nullstellensatz (Corollary 2.25):

Proposition 2.66 (Classification of closed and open subfunctors of affine schemes)**.** *Let A be a ring.*

(i) *The construction $F\mapsto \mathrm{V}(F)$ induces an order-reversing bijection*

$$\{\textit{ideals in } A\} \xrightarrow{\sim} \{\textit{closed subfunctors of } \mathrm{Spec}(A)\}.$$

(ii) *The construction $F\mapsto \mathrm{D}(F)$ induces an order-preserving bijection*

$$\{\textit{radical ideals in } A\} \xrightarrow{\sim} \{\textit{open subfunctors of } \mathrm{Spec}(A)\}.$$

Definition 2.67 (Closed and open immersions)**.** Let $f\colon Y\to X$ be a map of algebraic functors.

(i) f is a *closed immersion* or *closed embedding* if it is a monomorphism whose image is a closed subfunctor of X.
(ii) f is an *open immersion* or *open embedding* if it is a monomorphism whose image is an open subfunctor of X.

Definition 2.68 (Locally closed subfunctor, immersion)**.** Let X be an algebraic functor.

(i) A subfunctor $Y\subset X$ is *locally closed* if there exists an open subfunctor $U\subset X$ containing Y as a closed subfunctor.
(ii) A map $f\colon Y\to X$ is an *immersion* if it is a monomorphism whose image is a locally closed subfunctor of X.

Proposition 2.69 (Closure properties of immersions)**.**

(i) *Consider a commutative triangle of algebraic functors*

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & X. & \end{array}$$

If f and g are closed immersions, so is h. If h is a closed immersion and f is a monomorphism, then g is a closed immersion. The same holds for open immersions and for immersions.

(ii) *Consider a cartesian square of algebraic functors*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X. \end{array}$$

If f is a closed immersion, so is f'. The same holds for open immersions and for immersions.

2.8. **Zariski descent.** Let $(f_i)_{i\in I}$ be a family of element in a ring R that generates the unit ideal. Then the intersection of the vanishing loci $\mathrm{V}(f_i)$ is the empty scheme but, in general, it is not true that $\mathrm{Spec}(R)$ is the union of the nonvanishing loci $\mathrm{D}(f_i)$ (see Example 2.61). In this section, we will show that this becomes true if we compute the union in the category of affine schemes. Concretely, this means that a map $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ is uniquely determined by a family of maps $\mathrm{D}(f_i) \to \mathrm{Spec}(S)$ that agree on all the intersections $\mathrm{D}(f_i)\cap \mathrm{D}(f_j)$. This is one of the most important results in the foundations of algebraic geometry, which we will later recast as the statement that *affine schemes satisfy Zariski descent.*

Theorem 2.70 (Zariski descent for modules)**.** *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R generating the unit ideal.*

(i) (Descent for morphisms) *For any R-modules M and N, the diagram*
$$\mathrm{Map}(N, M) \to \prod_{i\in I} \mathrm{Map}(N_{f_i}, M_{f_i}) \rightrightarrows \prod_{i,j\in I} \mathrm{Map}(N_{f_if_j}, M_{f_if_j})$$
is an equalizer.

(ii) (Descent for objects) *Suppose given*
- *an R_{f_i}-module M_i for each $i \in I$ and*
- *an $R_{f_if_j}$-linear isomorphism $\alpha_{ij}\colon (M_i)_{f_j} \xrightarrow{\sim} (M_j)_{f_i}$ for each $(i,j) \in I^2$,*
- *such that $\alpha_{jk}\circ\alpha_{ij} = \alpha_{ik}\colon (M_i)_{f_jf_k} \xrightarrow{\sim} (M_k)_{f_if_j}$ for each $(i,j,k) \in I^3$.*

Then there exists an R-module M with R_{f_i}-linear isomorphisms $\beta_i\colon M_{f_i} \xrightarrow{\sim} M_i$ for all $i \in I$, such that $\alpha_{ij}\circ\beta_i = \beta_j\colon M_{f_if_j} \xrightarrow{\sim} (M_j)_{f_i}$ for all $(i,j) \in I^2$. Moreover, this data is unique up to unique isomorphism.

Taking $N = R$ in Theorem 2.70(i), we get the following special case:

Corollary 2.71. *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R generating the unit ideal. For any R-module M, the diagram*
$$M \to \prod_{i\in I} M_{f_i} \rightrightarrows \prod_{i,j\in I} M_{f_if_j}$$
is an equalizer in Mod_R.

Specializing further to $M = R$, we get the following result:

Corollary 2.72 (Zariski descent for affine schemes)**.** *Let R be a ring and let $(f_i)_{i\in I}$ be a family of elements of R generating the unit ideal. For any affine scheme X, the diagram*
$$X(R) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_if_j})$$
is an equalizer.

Remark 2.73. Since $X(R) \simeq \mathrm{Map}(\mathrm{Spec}(R), X)$, $\mathrm{Spec}(R_{f_i}) \simeq \mathrm{D}(f_i)$, and $\mathrm{Spec}(R_{f_if_j}) \simeq \mathrm{D}(f_i)\cap \mathrm{D}(f_j)$, Corollary 2.72 says that $\mathrm{Spec}(R)$ is the union of the open subschemes $\mathrm{D}(f_i)$ in Aff. More precisely, if $\mathrm{Glue}(I)$ is the poset with morphisms $i \leftarrow (i,j) \to j$ for all $i,j \in I$, then $\mathrm{Spec}(R)$ is the colimit of the diagram $\mathrm{Glue}(I) \to \mathrm{Aff}$ sending i to $\mathrm{D}(f_i)$ and (i,j) to $\mathrm{D}(f_i)\cap \mathrm{D}(f_j)$.

The following result is a generalization of Corollary 2.72, of which it is in fact a formal consequence:

Corollary 2.74 (Functions on nonvanishing loci)**.** *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R with image $F \subset R$. For any affine scheme X, there is an equalizer diagram*

$$\mathrm{Map}(\mathrm{D}(F), X) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j}).$$

Example 2.75. Using Corollary 2.74 with $X = \mathbb{A}^1$, we can easily compute that the inclusion $\mathbb{A}^I - 0 \hookrightarrow \mathbb{A}^I$ induces an isomorphism $\mathcal{O}(\mathbb{A}^I) \xrightarrow{\sim} \mathcal{O}(\mathbb{A}^I - 0)$ as soon as $|I| \geq 2$. Since $\mathcal{O}\colon \mathrm{Aff}^{\mathrm{op}} \to \mathrm{CAlg}$ is an equivalence of categories, this implies that $\mathbb{A}^I - 0$ is not an affine scheme. In particular, if $n \geq 2$, $\mathbb{A}^n - 0$ is an example of a quasi-affine scheme that is not affine.

Definition 2.76 (Zariski-local property)**.** A property P of modules (resp. of linear maps, of algebras, etc.) is *Zariski-local* if, for any ring R and family $(f_i)_{i\in I}$ generating the unit ideal in R, an R-module M (resp. an R-linear map $M \to N$, an R-algebra A, etc.) has property P if and only if, for each $i \in I$, the R_{f_i}-module M_{f_i} (resp. the R_{f_i}-linear map $M_{f_i} \to N_{f_i}$, the R_{f_i}-algebra A_{f_i}, etc.) has property P.

Proposition 2.77 (Examples of Zariski-local properties)**.** *The following properties of modules are Zariski-local:*

(i) *being zero,*
(ii) *finite generation,*
(iii) *finite presentation,*
(iv) *projectivity,*
(v) *flatness.*

The following properties of linear maps are Zariski-local:

(vi) *being zero,*
(vii) *injectivity,*
(viii) *surjectivity,*
(ix) *bijectivity.*

The following properties of algebras are Zariski-local:

(x) *finite generation,*
(xi) *finite presentation.*

The following properties of sequences of modules are Zariski-local:

(xii) *exactness.*

Remark 2.78. Further Zariski-local properties of modules are: being torsion, torsion-freeness. The following properties of modules are *not* Zariski-local: freeness, injectivity.

Proposition 2.79 (Zariski-local nature of immersions)**.** *Let R be a ring and let $u\colon X \to \mathrm{Spec}(R)$ be a map of algebraic functors. Suppose that X satisfies Zariski descent in the sense of Corollary 2.72. Let $(f_i)_{i\in I}$ generate the unit ideal in R, and let $u_i\colon X \times_{\mathrm{Spec}(R)} \mathrm{D}(f_i) \to \mathrm{D}(f_i)$ be the base change of u to $\mathrm{D}(f_i)$. For each of the following classes of maps, if each u_i belongs to the class, so does u:*

(i) *monomorphisms,*
(ii) *closed immersions,*
(iii) *open immersions,*
(iv) *immersions.*

Proposition 2.80. *Let X be a quasi-affine k-scheme and let $Z \hookrightarrow X$ be a closed immersion. Then Z is a quasi-affine k-scheme.*

2.9. **Finiteness properties.** Recall that a k-algebra is *of finite presentation* if it is isomorphic to $k[\Sigma]$ where Σ is a system of finitely many polynomial equations in finitely many variables, and it is *of finite type* if it is isomorphic to $k[\Sigma]$ where Σ has finitely many variables (but any number of equations). We denote the respective full subcategories of CAlg_k by $\mathrm{CAlg}_k^{\mathrm{fp}}$ and $\mathrm{CAlg}_k^{\mathrm{ft}}$. It turns out that these finiteness conditions can naturally be expressed in terms of the algebraic k-functor $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$.

Definition 2.81 (Locally of finite presentation/type)**.** Let $X\colon \mathrm{CAlg}_k \to \mathrm{Set}$ be an algebraic k-functor.

(i) X is *locally of finite presentation* if it preserves filtered colimits.
(ii) X is *locally of finite type* if it preserves the colimits of filtered diagrams with injective transition maps.

Proposition 2.82. *Let k be a ring and A a k-algebra.*

(i) *A is of finite presentation if and only if* $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ *is locally of finite presentation.*
(ii) *A is of finite type if and only if* $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ *is locally of finite type.*

Example 2.83.

(i) $\mathbb{A}^I_k$ is locally of finite type if and only if I is a finite set, in which case it is also locally of finite presentation. The same holds for the punctured affine spaces $\mathbb{A}^I_k - 0$.
(ii) The affine k-scheme $\mathbb{A}(M)$ is locally of finite presentation (resp. of finite type) if and only if the k-module M is of finite presentation (resp. of finite type).
(iii) The algebraic k-functor $\mathbb{A}^\vee(M)$ of Remark 2.38 is locally of finite presentation for any k-module M, since the tensor product preserves colimits in each variable.
(iv) The affine $\mathbb{Z}$-schemes $*$, $\varnothing$, Idem, $\mathbb{G}_\mathrm{m}$, $\mathbb{G}_\mathrm{a}$, Mat_n, GL_n, and SL_n from §2.4 are all locally of finite presentation.

Remark 2.84. Since filtered colimits commute with finite limits in the category of sets, the condition of being locally of finite presentation or of finite type is preserved by finite limits in $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$.

Remark 2.85 (Compatibility with base change)**.** Let $\varphi\colon k \to k'$ be a ring map. Since both the forgetful functor $\mathrm{CAlg}_{k'} \to \mathrm{CAlg}_k$ and its left adjoint $\varphi^*\colon \mathrm{CAlg}_k \to \mathrm{CAlg}_{k'}$ preserve filtered colimits, it follows from Corollary 2.42 that both base change along φ and Weil restriction along φ preserve the property of being locally of finite presentation or locally of finite type. On the other hand, the third functor $\varphi_\sharp$ usually does not. For example, $\mathrm{Spec}(\mathbb{C})$ is locally of finite presentation as an affine $\mathbb{R}$-scheme, but not as an affine $\mathbb{Q}$-scheme.

Recall that an algebraic k-functor can be thought of as a map of algebraic functors $X \to \mathrm{Spec}(k)$ (Corollary 2.48). Remark 2.85 implies that, for any cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(k), \end{array}$$

if f is locally of finite presentation or of finite type, so is f'. Consequently, we can extend Definition 2.81 to arbitrary maps of algebraic functors as follows:

Definition 2.86 (Morphism locally of finite presentation/type)**.** Let $f\colon X \to S$ be a map of algebraic functors.

(i) f is *locally of finite presentation* if, for every ring k and every k-point $\mathrm{Spec}(k) \to S$, the algebraic k-functor $X \times_S \mathrm{Spec}(k)$ is locally of finite presentation.
(ii) f is *locally of finite type* if, for every ring k and every k-point $\mathrm{Spec}(k) \to S$, the algebraic k-functor $X \times_S \mathrm{Spec}(k)$ is locally of finite type.

Concretely, under the equivalence of Corollary 2.48, the base change of $f\colon X \to S$ along $s\colon \mathrm{Spec}(k) \to S$ is the algebraic k-functor $\mathrm{CAlg}_k \to \mathrm{Set}$ given by

$$(\varphi\colon k \to R) \mapsto \{x \in X(R) \mid f(x) = s(\varphi) \text{ in } S(R)\} = \left\{ \begin{array}{ccc} \mathrm{Spec}(R) & \overset{x}{\dashrightarrow} & X \\ {\scriptstyle \mathrm{Spec}(\varphi)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(k) & \xrightarrow{s} & S \end{array} \right\}.$$

Example 2.87. Any open immersion is locally of finite presentation, and any closed immersion is locally of finite type. A closed immersion $i\colon Z \hookrightarrow X$ is locally of finite presentation if and only if, for every ring R and every $x\colon \mathrm{Spec}(R) \to X$, there exists a *finite* subset $F \subset R$ such that $x^{-1}(i(Z)) = \mathrm{V}(F)$.

Proposition 2.88 (Closure properties of morphisms locally of finite presentation/type)**.**

(i) *Consider a commutative triangle of algebraic functors*

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & X. & \end{array}$$

If f is locally of finite presentation, then g is locally of finite presentation if and only if h is. The same holds for "locally of finite type".

(ii) *Consider a cartesian square of algebraic functors*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X. \end{array}$$

If f is locally of finite presentation, so is f'. The same holds for "locally of finite type".

2.10. **The Nullstellensatz.** Consider a monic polynomial $f \in k[x]$ over a field k. By the functorial Nullstellensatz (Corollary 2.25), we know that f is determined by its zero sets in all k-algebras. On the other hand, by the elementary theory of fields, we know that f splits into linear factors over some finite field extension of k. Hence, if we know the zero sets of f over any finite field extension of k, then we know the original polynomial f provided it is separable (i.e., does not have multiple roots). In general, the zero sets of f over finite field extensions of k determine the *radical* of f, which is the product of the prime factors of f without multiplicity. The *Nullstellensatz* of Hilbert generalizes the latter statement to systems of polynomial equations in several variables: given $f_1, \dots, f_m \in k[x_1, \dots, x_n]$, the sets of common zeros of these polynomials in all finite field extensions of k determine the radical ideal $\sqrt{(f_1, \dots, f_n)}$. This nontrivial theorem was at the heart of classical algebraic geometry, which was only concerned with solutions of polynomial equations in fields. In this section, we review this result while also pointing out some shortcomings of the classical perspective.

For a ring k, define the maps

$$\{\text{subsets of } k[x_1, \dots, x_n]\} \underset{\mathrm{I}}{\overset{\mathrm{V}}{\rightleftarrows}} \{\text{subsets of } k^n\},$$

as follows:

$$\mathrm{V}(F) = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in F\},$$
$$\mathrm{I}(X) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

Note that both maps are order-reversing and that $F \subset \mathrm{I}(\mathrm{V}(F))$ and $X \subset \mathrm{V}(\mathrm{I}(X))$ (in other words, this is an adjunction between posets). Note also that $\mathrm{I}(X)$ is always an ideal in $k[x_1, \dots, x_n]$, and it is even a radical ideal if k is reduced (if a power of f vanishes on X, so does f). Call a subset $X \subset k^n$ *algebraic* if it lies in the image of V, or equivalently if $X = \mathrm{V}(\mathrm{I}(X))$.

Theorem 2.89 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and let $n \in \mathbb{N}$. For any subset $F \subset k[x_1, \dots, x_n]$, we have*

$$\mathrm{I}(\mathrm{V}(F)) = \sqrt{(F)}.$$

Consequently, the maps V *and* I *define a one-to-one correspondence*

$$\{\textit{radical ideals in } k[x_1, \dots, x_n]\} \underset{\mathrm{I}}{\overset{\mathrm{V}}{\underset{\sim}{\rightleftarrows}}} \{\textit{algebraic subsets of } k^n\}.$$

We can upgrade this result to an equivalence of categories as follows. Define the category AffSet_k of *affine algebraic sets* over k as follows:

- An object of AffSet_k is a pair (n, X) with $n \geq 0$ and $X \subset k^n$ an algebraic subset.
- A morphism $(n, X) \to (m, Y)$ is a map $f: X \to Y$ such that there exists a polynomial map $k^n \to k^m$ extending f.

Recall that a ring R is *reduced* if 0 is the only nilpotent element of R. We denote by $\mathrm{CAlg}_k^{\mathrm{red}}$ the category of reduced k-algebras.

Corollary 2.90. *Let k be an algebraically closed field. Then there is an equivalence of categories*

$$\mathrm{AffSet}_k \xrightarrow{\sim} (\mathrm{CAlg}_k^{\mathrm{ft,red}})^{\mathrm{op}},$$
$$(n, X) \mapsto k[x_1, \dots, x_n]/\mathrm{I}(X).$$

Hence, AffSet_k *is equivalent to the full subcategory of* Aff_k *spanned by the reduced affine k-schemes of finite type.*

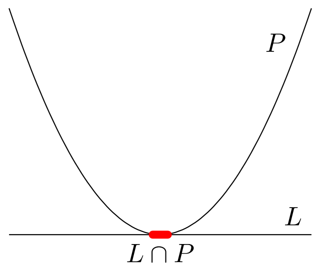
Remark 2.91. By Hilbert's basis theorem, "finite type" and "finite presentation" are equivalent for algebras over a field (and more generally over a noetherian ring).

We can also formulate a Nullstellensatz for an arbitrary field k as follows. Denote by $\mathrm{Field}_k^{\mathrm{fin}} \subset \mathrm{CAlg}_k$ the full subcategory of finite field extensions of k.

Corollary 2.92. *Let k be a field and let $n \in \mathbb{N}$. Then there is an order-reversing bijection*

$$\mathrm{V}\colon \{\textit{radical ideals in } k[x_1,\ldots,x_n]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbb{A}^n\colon \mathrm{Field}_k^{\mathrm{fin}} \to \mathrm{Set}\}.$$

Example 2.93 (Non-reduced intersections)**.** Even in the context of algebraic geometry over an algebraically closed field k, there are geometric phenomena that are not captured by only considering solutions in k. Consider for examples the vanishing loci $L = \mathrm{V}(y)$ and $P = \mathrm{V}(y - x^2)$ in $\mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$. Since the k-algebras $k[x,y]/(y) \simeq k[t]$ and $k[x,y]/(y-x^2) \simeq k[t]$ are reduced, these affine k-schemes are determined by the algebraic sets $L(k)$ and $P(k)$ in k^2 (by the Nullstellensatz). The intersection $L(k) \cap P(k)$ is the algebraic set $\{(0,0)\} \subset k^2$, which in turn corresponds to the subfunctor $\mathrm{V}(x,y) \subset \mathbb{A}^2_k$. However, the functorial intersection $L \cap P \subset \mathbb{A}^2_k$ is the subfunctor $\mathrm{V}(x^2, y)$, which is isomorphic to $\mathrm{Spec}(k[t]/(t^2))$. One can think of $\mathrm{V}(x^2,y)$ as a first-order infinitesimal neighborhood of the origin $\mathrm{V}(x,y)$ along the x-axis; this captures the fact that the line L is tangent (to first order) to the parabola P, so that they both contain the same infinitesimal horizontal segment at the origin. This residual tangency information in the intersection can only be seen by evaluating the functor $L \cap P$ on non-reduced k-algebras.



This also resolves another issue in classical algebraic geometry, which is that intersections do not vary nicely in families. Consider for example the family of horizontal line $L_a = \mathrm{V}(y-a)$ for $a \in k$. The intersection $L_a(k) \cap P(k)$ has exactly two points for any $a \neq 0$ (since k is algebraically closed), but only a single point when $a = 0$. On the other hand, the scheme-theoretic intersection $L_a \cap P$ is given by a 2-dimensional k-algebra for *all* $a \in k$, namely $k \times k$ when $a \neq 0$ and $k[t]/(t^2)$ when $a = 0$.

Example 2.94 (Geometry in mixed characteristic)**.** Another aspect that is not captured by classical algebraic geometry over fields is algebraic geometry in *mixed characteristic*, i.e., involving rings R that do not contain any field. This is especially relevant in number theory, which studies rings of integers in finite extensions of $\mathbb{Q}$. Such rings can map to fields with different characteristics, which sometimes allows us to transport results from one characteristic to another. As a very basic example, consider the following proof that $\sqrt{2}$ is irrational (which is a reformulation of the usual argument). A positive rational number x such that $x^2 = 2$ is the same thing as an element of $X(\mathbb{Z})$ where $X \subset \mathbb{P}^1$ is the solution functor to the homogeneous polynomial equation $x^2 = 2y^2$. Since X is a functor, the ring map $\mathbb{Z} \to \mathbb{Z}/4$ induces a map $X(\mathbb{Z}) \to X(\mathbb{Z}/4)$. Since the squares in $\mathbb{Z}/4$ are 0 and 1, none of the six elements of $\mathbb{P}^1(\mathbb{Z}/4)$ satisfy the equation $x^2 = 2y^2$, so that $X(\mathbb{Z}/4)$ is empty. It follows that $X(\mathbb{Z})$ is also empty, i.e., that there does not exist $x \in \mathbb{Q}$ with $x^2 = 2$.

3. Projective geometry

3.1. **Projective spaces over a field.**

Let k be a field. The classical projective n-space over k is the set of lines through the origin in k^{n+1}:

$$\mathbb{P}^n(k) = \{\text{1-dimensional subspaces of } k^{n+1}\}.$$

Given a nonzero $(n+1)$-tuple $(a_0,\ldots,a_n) \in k^{n+1} - \{0\}$, we denote by $[a_0 : \ldots : a_n]$ the 1-dimensional subspace of k^{n+1} containing $(a_0,\ldots,a_n)$. This identifies $\mathbb{P}^n(k)$ with the set of orbits of the (free) action of $k^\times$ on $k^{n+1} - \{0\}$ by scalar multiplication:

$$(k^{n+1} - \{0\})/k^\times \xrightarrow{\sim} \mathbb{P}^n(k),$$
$$(a_0,\ldots,a_n) \mapsto [a_0 : \ldots : a_n].$$

The set $\mathbb{P}^n(k)$ is the union of the $n+1$ subsets $U_0,\ldots,U_n$, where

$$U_i = \{[a_0 : \ldots : a_n] \in \mathbb{P}^n(k) \mid a_i \text{ is a unit}\}.$$

Each U_i can be identified with $\mathbb{A}^n(k) = k^n$ via

$$\begin{aligned} U_i &\simeq \mathbb{A}^n(k), \\ [a_0 : \dots : a_n] &\mapsto \left(\frac{a_0}{a_i}, \dots, \widehat{\frac{a_i}{a_i}}, \dots, \frac{a_n}{a_i}\right), \\ [a_0 : \dots : 1 : \dots : a_n] &\leftarrow\!\shortmid (a_0, \dots, \widehat{a_i}, \dots, a_n). \end{aligned}$$

The complement $H_i = \mathbb{P}^n(k) - U_i$ is given by

$$H_i = \{[a_0 : \dots : a_n] \in \mathbb{P}^n(k) \mid a_i = 0\}$$

and can be identified with $\mathbb{P}^{n-1}(k)$ by dropping the ith coordinate. We often think of $\mathbb{P}^n(k)$ as the completion of $U_0 = \mathbb{A}^n(k)$ obtained by adding a point "at infinity" on every line through the origin. These points at infinity form the *hyperplane at infinity* $H_0 = \mathbb{P}^{n-1}(k)$ in $\mathbb{P}^n(k)$.

Since the k-vector space k^{n+1} is canonically self-dual, we can identify $\mathbb{P}^n(k)$ with the set of 1-dimensional *quotient spaces* of k^{n+1}:

$$\begin{aligned} \mathbb{P}^n(k) &\simeq \{\text{1-dimensional quotient spaces of } k^{n+1}\}, \\ L &\mapsto k^{n+1}/L^\perp. \end{aligned}$$

Concretely, the line $[a_0 : \dots : a_n]$ corresponds to the coimage of the map $(a_0, \dots, a_n) \colon k^{n+1} \twoheadrightarrow k$.

3.2. **Vector spaces and lines.** Our goal is to define the set $\mathbb{P}^n(R)$ for an arbitrary ring R. To that end, we need to make sense of *lines* over R. In this section, we define vector spaces and lines over rings, as well as subspaces and quotient spaces.

Proposition 3.1. *Let R be a ring. For an R-module V, the following conditions are equivalent:*

(i) *V is finitely generated and projective;*
(ii) *V is finitely presented and flat;*
(iii) *V is a direct summand of R^n for some $n \in \mathbb{N}$;*
(iv) *there exist elements $f_1, \dots, f_n \in R$ generating the unit ideal such that each V_{f_i} is a finitely generated free R_{f_i}-module;*
(v) *V is* dualizable, *i.e., there exists $V' \in \mathrm{Mod}_R$ and a map $e \colon V \otimes_R V' \to R$ such that $e \otimes (-)$ exhibits $V \otimes_R (-)$ as left adjoint to $V' \otimes_R (-)$.*

Definition 3.2 (Vector space)**.** An R-module satisfying the equivalent conditions of Proposition 3.1 is called a *vector space* over R.[3] We denote by $\mathrm{Vect}_R \subset \mathrm{Mod}_R$ the full subcategory of vector spaces.

Recall that any module M over a field admits a basis and that all bases have the same cardinality, which is called the *dimension* of M. To extend this notion to modules over a ring R, we consider *R-fields*, i.e., R-algebras that are also fields:

Definition 3.3 (Rank of a module)**.** Let R be a ring, M an R-module, and κ an R-field. The *rank* of M at κ, denoted by $\mathrm{rk}_\kappa(M)$, is the dimension of the κ-module $M \otimes_R \kappa$. We say that M has *constant rank r* if $\mathrm{rk}_\kappa(M) = r$ for every R-field κ.

Remark 3.4 (Residue fields)**.** Let R be a ring. Recall that the *residue field* of R at a prime ideal $\mathfrak{p} \subset R$ is the R-field

$$\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \simeq \mathrm{Frac}(R/\mathfrak{p}).$$

The residue fields of R form an initial family of objects of the category Field_R in the following sense: for every R-field κ, there exists a unique prime ideal $\mathfrak{p}$ (namely the kernel of $R \to \kappa$) such that $R \to \kappa$ factors through $\kappa(\mathfrak{p})$, and the factorization is unique. In other words, the category of R-fields decomposes as

$$\mathrm{Field}_R = \coprod_{\mathfrak{p} \subset R} \mathrm{Field}_{\kappa(\mathfrak{p})}.$$

Since extension of scalars between fields preserves dimension, we have $\mathrm{rk}_\kappa(M) = \mathrm{rk}_{\kappa(\mathfrak{p})}(M)$ for any κ in the summand indexed by $\mathfrak{p}$.

[3]This definition is nonstandard: when k is a field, every k-module is traditionally called a "vector space", but only the finite-dimensional ones satisfy this definition.

Remark 3.5 (Short exact sequences of vector spaces)**.** Since vector spaces are projective, every short exact sequence of vector spaces splits. Hence, for any ring map $\varphi\colon R \to R'$, the functor $\varphi^*\colon \mathrm{Vect}_R \to \mathrm{Vect}_{R'}$ preserves short exact sequences. In particular, if

$$0 \to U \to V \to W \to 0$$

is a short exact sequence in Vect_R and κ is any R-field, then $\mathrm{rk}_\kappa(V) = \mathrm{rk}_\kappa(U) + \mathrm{rk}_\kappa(W)$.

Remark 3.6 (Modules of constant rank 0)**.** By Propositions 3.1(iv) and 2.77(i), a vector space is zero if and only if it has constant rank 0. Note that this is not true for more general modules. For example, the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ has constant rank 0.

Proposition 3.7. *Let R be a ring, For an R-module L, the following conditions are equivalent:*

(i) *L is a vector space of constant rank* 1*.*
(ii) *L is* invertible*, i.e., there exists $L' \in \mathrm{Mod}_R$ such that $L \otimes_R L' \simeq R$.*

Definition 3.8 (Line)**.** An R-module satisfying the equivalent conditions of Proposition 3.7 is called a *line* over R. We denote by $\mathrm{Line}_R \subset \mathrm{Vect}_R$ the full subcategory of lines. A line is called *trivial* if it is isomorphic to R.

Definition 3.9 (Universally injective map)**.** Let R be a ring and let M and N be R-modules. An R-linear map $f\colon M \to N$ is called *universally injective* if, for every ring map $R \to S$, the map $f \otimes_R S\colon M \otimes_R S \to N \otimes_R S$ is injective.

Remark 3.10.

(i) If the R-linear map $f\colon M \to N$ is universally injective, then for every R-module P, $f \otimes_R P\colon M \otimes_R P \to N \otimes_R P$ is injective. This follows from applying the definition of universal injectivity to the ring map $R \to \mathrm{Sym}_R(P)$ and using that P is a direct summand of $\mathrm{Sym}_R(P)$.
(ii) We need not consider the dual notion of *universally surjective* map, since surjective maps are preserved by base change. In other words, every surjective map is universally surjective.

Example 3.11.

(i) Any map with a retraction (equivalently, any direct summand inclusion) is universally injective.
(ii) The inclusion $\bigoplus_{\mathbb{N}} \mathbb{Z} \hookrightarrow \prod_{\mathbb{N}} \mathbb{Z}$ is an example of a universally injective map of abelian groups that does not admit a retraction.
(iii) Let $f \in R$. The "multiplication by f" map $R \to R$ is injective if and only if f is not a zero divisor, but it is universally injective if and only if f is a unit (as it becomes the zero map after base change to R/f).

Remark 3.12. By Proposition 2.77, "vector space" and "line" are Zariski-local properties of modules, and "universally injective" is a Zariski-local property of linear maps. In fact, by Proposition 3.1(iv), "vector space" and "line" are precisely the minimal weakenings of the properties "free of finite rank" and "free of rank 1" that are Zariski-local.

Definition 3.13 (Subspace and quotient space)**.** Let V be an R-module.

(i) A *subspace* of V is a submodule $U \subset V$ such that U is a vector space and the inclusion map $U \hookrightarrow V$ is universally injective.
(ii) A *quotient space* of V is a quotient module $V \twoheadrightarrow W$ such that W is a vector space.

This definition is mostly used when V itself is a vector space. In this case, there are several useful characterizations of subspaces and of quotient spaces:

Proposition 3.14 (Characterization of subspaces)**.** *Let R be a ring and let U and V be vector spaces over R. For a map $f\colon U \to V$, the following are equivalent:*

(i) *f is universally injective.*
(ii) *For every R-field κ, $f \otimes_R \kappa$ is injective.*
(iii) *f is injective and the cokernel of f is a vector space.*
(iv) *f admits a retraction.*
(v) *The dual map $f^\vee\colon V^\vee \to U^\vee$ is surjective.*

Proposition 3.15 (Characterization of quotient spaces)**.** *Let R be a ring and let V and W be vector spaces over R. For a map $f\colon V \to W$, the following are equivalent:*

(i) *f is surjective.*
(ii) *For every R-field κ, $f \otimes_R \kappa$ is surjective.*

(iii) *f is surjective and the kernel of f is a vector space.*
(iv) *f admits a section.*
(v) *The dual map $f^\vee\colon W^\vee \to V^\vee$ is universally injective.*

Corollary 3.16. *Let R be a ring and let V and W be vector spaces over R. A map $f\colon V \to W$ is an isomorphism if and only if, for every R-field κ, $f \otimes_R \kappa$ is an isomorphism.*

Corollary 3.17. *Let R be a ring and let V and W be vector spaces over R such that $\mathrm{rk}_\kappa(V) = \mathrm{rk}_\kappa(W)$ for all R-fields κ. For a map $f\colon V \to W$, the following are equivalent:*
(i) *f is an isomorphism;*
(ii) *f is surjective;*
(iii) *f is universally injective.*

Corollary 3.18. *Let V be a vector space over R. Duality induces bijections*

$$\{\textit{subspaces of } V\} \simeq \{\textit{quotient spaces of } V^\vee\}, \qquad \{\textit{quotient spaces of } V\} \simeq \{\textit{subspaces of } V^\vee\},$$
$$(U \hookrightarrow V) \mapsto (V^\vee \twoheadrightarrow U^\vee), \qquad (V \twoheadrightarrow W) \mapsto (W^\vee \hookrightarrow V^\vee).$$

Remark 3.19. In practice, we do not distinguish between *submodules* of M and *monomorphisms* into M, nor between *quotient modules* of M and *epimorphisms* out of M: we may always identify a monomorphism with its image and an epimorphism with its coimage. Two monomorphisms into M have the same image if and only if they are isomorphic over M (in which case the isomorphism is unique). Similarly, two epimorphisms out of M have the same coimage if and only if they are (uniquely) isomorphic under M. These identifications are happening for example in the statement of Corollary 3.18.

Warning 3.20. Let $f\colon V \to W$ be a map between vector spaces. In general, neither $\ker(f)$ nor $\mathrm{coker}(f)$ is a vector space. By Propositions 3.14 and 3.15, $\ker(f)$ is a vector space if f is surjective, and $\mathrm{coker}(f)$ is a vector space if f is universally injective (but injectivity does not suffice).

3.3. Projective spaces.

Definition 3.21 (Projective space)**.** Let k be a ring and I a set. The *projective I-space* over k is the algebraic k-functor

$$\mathbb{P}^I_k\colon \mathrm{CAlg}_k \to \mathrm{Set}, \quad R \mapsto \{\text{quotient lines of } R^{(I)}\}.$$

We simply write $\mathbb{P}^I$ when $k = \mathbb{Z}$. For $n \geq -1$, the *projective n-space* over k is $\mathbb{P}^n_k = \mathbb{P}^{\{0,\dots,n\}}_k$. It is also called the *projective line* if $n = 1$ and the *projective plane* if $n = 2$.

Remark 3.22.
(i) $\mathbb{P}^{-1}_k = \mathbb{P}^\varnothing_k$ is the empty k-scheme $\varnothing$ (Example 2.30).
(ii) $\mathbb{P}^0_k$ is the final k-scheme $\mathrm{Spec}(k)$.
(iii) If I is a *finite* set, Corollary 3.18 provides a natural bijection

$$\mathbb{P}^I_k(R) \simeq \{\text{sublines of } R^I\}.$$

However, if I is infinite or more generally if we replace the R-module $R^{(I)}$ by some R-module that is not a vector space (see Example 3.42), then it is necessary to use quotient lines instead of sublines in the definition.

Notation 3.23 (Projective coordinates)**.** Let L be a line over R and let $(a_0, \dots, a_n) \in L^{n+1}$ be a family generating L as an R-module. The induced map $R^{n+1} \to L$ is then surjective and its coimage is a quotient line of R^{n+1}, which we denote by

$$[a_0 : \ldots : a_n] \in \mathbb{P}^n(R).$$

Given another line M and generating family $(b_0, \dots, b_n) \in M$, we have $[a_0 : \ldots : a_n] = [b_0 : \ldots : b_n]$ if and only if there is a (necessarily unique) isomorphism $L \xrightarrow{\sim} M$ sending each a_i to b_i.

The affine and projective spaces $\mathbb{A}^I_k$ and $\mathbb{P}^I_k$ are related via the punctured affine space $\mathbb{A}^I_k - 0$ of Example 2.57. Recall that $\mathbb{A}^I_k - 0$ is a subfunctor of $\mathbb{A}^I_k$ given by

$$(\mathbb{A}^I_k - 0)(R) = \{a \in R^I \mid (a) = R\}.$$

Under the identification of R^I with the R-linear dual of $R^{(I)}$, I-tuples generating the unit ideal correspond to surjective maps $R^{(I)} \twoheadrightarrow R$. In particular, there is a canonical map $\mathbb{A}^I_k - 0 \to \mathbb{P}^I_k$ sending an I-tuple to the corresponding quotient of $R^{(I)}$. We therefore have a zigzag of morphisms

$$\mathbb{A}^I_k \hookleftarrow \mathbb{A}^I_k - 0 \to \mathbb{P}^I_k$$

relating the affine space and the projective space. For $n \geq -1$, this specializes to a zigzag

$$\mathbb{A}^{n+1}_k \hookleftarrow \mathbb{A}^{n+1}_k - 0 \to \mathbb{P}^n_k.$$

Proposition 3.24. *Let I be a set and R a ring. The map $\mathbb{A}^I - 0 \to \mathbb{P}^I$ induces an injective map*

$$(\mathbb{A}^I - 0)(R)/R^\times \hookrightarrow \mathbb{P}^I(R),$$

where $R^\times$ acts on $(\mathbb{A}^I - 0)(R)$ by scalar multiplication, whose image consists exactly of the trivial *quotient lines of $R^{(I)}$. In particular, if every line over R is trivial (e.g., if R is a local ring or a principal ideal domain), then this map is bijective.*

Remark 3.25. The scaling action of $R^\times$ on $(\mathbb{A}^I - 0)(R)$ is functorial in R, i.e., it is an action of the affine group scheme $\mathbb{G}_\mathrm{m}$ on the algebraic functor $\mathbb{A}^I - 0$. By Proposition 3.24, the map $\mathbb{A}^I - 0 \to \mathbb{P}^I$ induces a monomorphism of algebraic functors $(\mathbb{A}^I - 0)/\mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{P}^I$.

Remark 3.26.

(i) Unlike the affine space construction $I \mapsto \mathbb{A}^I$, the constructions $I \mapsto \mathbb{P}^I$ and $I \mapsto \mathbb{A}^I - 0$ are not contravariantly functorial on the whole category of sets, but only on the subcategory of sets and *surjections*.

(ii) A technical difference between affine and projective spaces is that $\mathbb{A}^I$ is quasi-compact for all I (this property will be defined in §7.3), whereas $\mathbb{P}^I$ and $\mathbb{A}^I - 0$ are quasi-compact if and only if I is finite. For this reason, many basic results in projective geometry assume a finite number of variables.

3.4. **Graded rings.**

Definition 3.27 (Homogeneous polynomial)**.** Let k be a ring, I a set, and $d \in \mathbb{N}$. A polynomial in $k[x_i \mid i \in I]$ is *homogeneous of degree d* if it is a k-linear combination of monomials of the form $\prod_{i\in I} x_i^{n_i}$ with $\sum_{i\in I} n_i = d$. We denote by $k[x_i \,|\, i \in I]_d \subset k[x_i \,|\, i \in I]$ the k-submodule of homogeneous polynomials of degree d.

Note that there is a direct sum decomposition

$$k[x_i \mid i \in I] = \bigoplus_{d\in\mathbb{N}} k[x_i \mid i \in I]_d,$$

which is compatible with the multiplication in the sense that the product of a homogeneous polynomial of degree d with one of degree e gives a homogeneous polynomial of degree $d + e$. These properties are abstracted in the notion of *graded ring*:

Definition 3.28 (Graded ring)**.** Let $(\Gamma, +, 0)$ be a commutative monoid (typically $\mathbb{N}$ or $\mathbb{Z}$). A Γ-*graded ring* is a ring A together with subgroups $A_\gamma \subset A$ for all $\gamma \in \Gamma$ such that:

(i) A is the direct sum of the subgroups A_γ: $\bigoplus_{\gamma\in\Gamma} A_\gamma \xrightarrow{\sim} A$;

(ii) $1 \in A_0$, and for all $\gamma, \delta \in \Gamma$, $A_\gamma A_\delta \subset A_{\gamma+\delta}$.

Elements of A_γ are called *homogeneous of degree γ*. If k is a ring, a Γ-*graded k-algebra* is a Γ-graded ring A with a ring map $k \to A$ that lands in A_0.

Definition 3.29 (Graded module)**.** Let A be a Γ-graded ring. A Γ-*graded A-module* is an A-module M together with subgroups $M_\gamma \subset M$ for all $\gamma \in \Gamma$ such that:

(i) M is the direct sum of the subgroups M_γ: $\bigoplus_{\gamma\in\Gamma} M_\gamma \xrightarrow{\sim} M$;

(ii) For all $\gamma, \delta \in \Gamma$, $A_\gamma M_\delta \subset M_{\gamma+\delta}$.

An ideal $I \subset A$ is called *homogeneous* if it is a Γ-graded A-module with $I_\gamma = I \cap A_\gamma$.

Remark 3.30. Let A be a Γ-graded ring and M a Γ-graded A-module.

(i) A_0 is a subring of A, each A_γ is an A_0-module, and each M_γ is an A_0-module.

(ii) Let $\Gamma \to \Delta$ be a map of commutative monoids. Then A is also a Δ-graded ring with $A_\delta = \bigoplus_{\gamma\mapsto\delta} A_\gamma$, and M is similarly a Δ-graded A-module. For example, any $\mathbb{N}$-graded ring A is a $\mathbb{Z}$-graded ring with $A_n = 0$ for $n < 0$.

(iii) An ideal in A is homogeneous if and only if it is generated by homogeneous elements.

(iv) If $I \subset A$ is a homogeneous ideal, the quotient ring A/I inherits a unique Γ-grading such that $A \twoheadrightarrow A/I$ is a graded map. Moreover, the A/I-module M/IM has a unique Γ-grading such that $M \to M/IM$ is a graded map.

(v) Let $S \subset A$ be a set of homogeneous elements whose degrees are invertible in Γ. Then the localized ring $A[S^{-1}]$ inherits a unique Γ-grading such that $A \to A[S^{-1}]$ is a graded map. Moreover, the $A[S^{-1}]$-module $M[S^{-1}]$ has a unique Γ-grading such that $M \to M[S^{-1}]$ is a graded map.

Example 3.31. Let k be a ring.

(i) For any set I, the polynomial algebra $k[x_i \mid i \in I]$ is an $\mathbb{N}$-graded k-algebra with $k[x_i \mid i \in I]_d$ the subgroup of homogeneous polynomials of degree d.
(ii) The Laurent polynomial algebra $k[x^{\pm 1}]$ is a $\mathbb{Z}$-graded k-algebra with x in degree 1. More generally, inverting any set of homogeneous polynomials in $k[x_i \mid i \in I]$ yields a $\mathbb{Z}$-graded k-algebra.
(iii) Let M be a k-module. The symmetric algebra $\mathrm{Sym}_k(M)$ has a canonical $\mathbb{N}$-grading

$$\mathrm{Sym}_k(M) = \bigoplus_{d \in \mathbb{N}} \mathrm{Sym}^d_k(M),$$

where $\mathrm{Sym}^d_k(M)$ is the dth symmetric power of M. This recovers (i) for $M = k^{(I)}$.
(iv) Let L be a line over k. Then $\bigoplus_{d \in \mathbb{Z}} L^{\otimes d}$ is naturally a $\mathbb{Z}$-graded k-algebra.

Notation 3.32. Let A be a Γ-graded ring, M a Γ-graded A-module, and $\gamma \in \Gamma$.

(i) We denote by $M(\gamma)$ the Γ-graded A-module whose underlying A-module is M with $M(\gamma)_\delta = M_{\gamma+\delta}$. This defines an endofunctor $M \mapsto M(\gamma)$, which is an equivalence if γ is invertible in Γ.
(ii) We denote by $A^{(\gamma)}$ the $\mathbb{N}$-graded ring $\bigoplus_{n \in \mathbb{N}} A_{n\gamma}$ with $(A^{(\gamma)})_n = A_{n\gamma}$. Similarly, $M^{(\gamma)}$ is an $\mathbb{N}$-graded $A^{(\gamma)}$-module.
(iii) Suppose γ invertible in Γ. Given $f \in A_\gamma$, A_f is a Γ-graded ring and we write $A_{(f)} = (A_f)_0$. Similarly, $M_{(f)} = (M_f)_0$ is an $A_{(f)}$-module. Note that there is an isomorphism of $\mathbb{Z}$-graded rings

$$A_{(f)}[x^{\pm 1}] \xrightarrow{\sim} A_f^{(\gamma)}, \quad x \mapsto f.$$

3.5. **Homogeneous polynomial equations.**

Definition 3.33 (System of homogeneous polynomial equations)**.** Let k be a ring and let I and J be sets. A *system of J homogeneous polynomial equations in I variables* over k is a J-tuple $\Sigma = (f_j)_{j \in J}$ of homogeneous polynomials in $k[x_i \mid i \in I]$. We denote by (Σ) the homogeneous ideal in $k[x_i \mid i \in I]$ generated by $(f_j)_{j \in J}$ and by $k[\Sigma]$ the $\mathbb{N}$-graded k-algebra $k[x_i \mid i \in I]/(\Sigma)$.

The point of considering homogeneous polynomial equations is that they have well-defined solutions in projective space, as we now explain.

Remark 3.34 (Symmetric powers of lines)**.** If L is a line over a ring R, the canonical quotient map $L^{\otimes d} \to \mathrm{Sym}^d_R(L)$ is an isomorphism. This follows from the observation that a d-linear map $L^{\times d} \to M$ is automatically symmetric.

Construction 3.35 (Evaluation of homogeneous polynomials)**.** Let k be a ring, I a set, $d \in \mathbb{N}$, R a k-algebra, and L a line over R. We construct an evaluation map

$$k[x_i \mid i \in I]_d \times L^I \to L^{\otimes d}, \quad (f, a) \mapsto f(a),$$

as follows. An element $a \in L^I$ is equivalently an R-linear map $a\colon R^{(I)} \to L$. The map $f \mapsto f(a)$ is then the composite

$$k[x_i \mid i \in I]_d = \mathrm{Sym}^d_k(k^{(I)}) \to \mathrm{Sym}^d_R(R^{(I)}) \xrightarrow{\mathrm{Sym}^d_R(a)} \mathrm{Sym}^d_R(L) = L^{\otimes d}.$$

Concretely, this is the unique k-linear map sending a degree d monomial $\prod_i x_i^{n_i}$ to the tensor product $\bigotimes_i a_i^{\otimes n_i} \in L^{\otimes d}$. Note that this recovers the usual evaluation when $L = R$, in which case also $L^{\otimes d} = R$.

Definition 3.36 (Vanishing locus)**.** Let $F \subset k[x_i \mid i \in I]$ be a set of homogeneous polynomials. The *vanishing locus* of F in $\mathbb{P}^I_k$ is the subfunctor $\mathrm{V}(F) \subset \mathbb{P}^I_k$ given by

$$\mathrm{V}(F)(R) = \{a\colon R^{(I)} \twoheadrightarrow L \mid f(a) = 0 \text{ for all } f \in F\} \subset \mathbb{P}^I(R)$$

This is indeed a subfunctor: for any k-algebra map $R \to S$, the induced map $\mathbb{P}^I(R) \to \mathbb{P}^I(S)$ sends $\mathrm{V}(F)(R)$ to $\mathrm{V}(F)(S)$.

It is clear that the subfunctor $\mathrm{V}(F) \subset \mathbb{P}^I_k$ depends only on the homogeneous ideal (F). However, unlike in the affine case, different homogeneous ideals can still have the same vanishing locus; we will discuss this in §3.8.

Definition 3.37 (Solution functor)**.** Let $\Sigma = (f_j)_{j\in J}$ be a system of J homogeneous polynomial equations in I variables over k. Its *solution functor* $\mathrm{hSol}_\Sigma\colon \mathrm{CAlg}_k \to \mathrm{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbb{P}^I_k$:

$$\mathrm{hSol}_\Sigma = \mathrm{V}(\{f_j \mid j \in J\}) \subset \mathbb{P}^I_k.$$

Definition 3.38 (Projective scheme)**.** A functor $\mathrm{CAlg}_k \to \mathrm{Set}$ is a *projective k-scheme* if it is isomorphic to hSol_Σ for some system of homogeneous polynomial equations Σ in *finitely many* variables over k. We denote by $\mathrm{Proj}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ the full subcategory spanned by the projective k-schemes.

Remark 3.39. The finiteness condition in the definition of projective k-scheme is a matter of convenience. It implies for example that any projective k-scheme is locally of finite type (see Definition 2.81). We will later see that projective k-schemes are *proper*, and for this the finiteness condition is essential.

Next, we want to show that the solution functor hSol_Σ depends only on the $\mathbb{N}$-graded k-algebra $k[\Sigma]$, analogously to $\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma])$ (Lemma 2.22). We first note that not every $\mathbb{N}$-graded k-algebra A is isomorphic to one of the form $k[\Sigma]$: this is the case if and only if A is generated by A_1 as a k-algebra, i.e., if and only if the canonical map $\mathrm{Sym}_k(A_1) \to A$ is surjective. The following construction will be generalized to arbitrary $\mathbb{N}$-graded algebras in §3.7, but the case of algebras generated in degree 1 is by far the most important case in practice.

Construction 3.40 (Proj of $\mathbb{N}$-graded algebras generated in degree 1)**.** Let k be a ring and A an $\mathbb{N}$-graded k-algebra generated by A_1. We define the algebraic k-functor $\mathrm{Proj}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ by

$$\mathrm{Proj}(A)(R) = \{\text{quotient lines } A_1 \otimes_k R \twoheadrightarrow L \text{ such that } \mathrm{Sym}_k(A_1) \to \mathrm{Sym}_R(L) \text{ factors through } A\}.$$

Proposition 3.41. *Let Σ be a system of homogeneous polynomial equations over k. Then there is a canonical isomorphism*

$$\mathrm{hSol}_\Sigma \simeq \mathrm{Proj}(k[\Sigma])\colon \mathrm{CAlg}_k \to \mathrm{Set}.$$

Hence, an algebraic k-functor X is a projective k-scheme if and only if $X \simeq \mathrm{Proj}(A)$ for some $\mathbb{N}$-graded k-algebra A that is generated by a finite subset of A_1.

Example 3.42 (The projective space of a module)**.** Let k be a ring and M a k-module. Consider the functor $\mathbb{P}(M)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ defined by

$$\mathbb{P}(M)(R) = \{\text{quotient lines of } M \otimes_k R\}.$$

If $\mathrm{Sym}_k(M)$ is the free k-algebra on M with its natural grading, we have $\mathbb{P}(M) \simeq \mathrm{Proj}(\mathrm{Sym}_k(M))$. In particular, if M is finitely generated, then $\mathbb{P}(M)$ is a projective k-scheme. The projective I-space is a special case of this construction: $\mathbb{P}^I_k \simeq \mathbb{P}(k^{(I)})$.

Analogously to Remark 2.38, we can also define a "dual" functor $\mathbb{P}^\vee(M)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ by

$$\mathbb{P}^\vee(M)(R) = \{\text{sublines of } M \otimes_k R\}.$$

If M is a vector space, then $\mathbb{P}^\vee(M^\vee) \simeq \mathbb{P}(M)$. In general, however, $\mathbb{P}^\vee(M)$ is not a scheme.

Remark 3.43. If A is an $\mathbb{N}$-graded k-algebra generated by A_1, then $\mathrm{Proj}(A)$ is a subfunctor of $\mathbb{P}(A_1)$.

3.6. **Loci associated with linear maps.** Consider the projective n-space $\mathbb{P}^n = \mathrm{Proj}(\mathbb{Z}[x_0, \dots, x_n])$, whose R-points are quotient lines $R^{\{0,\dots,n\}} \twoheadrightarrow L$. Given a homogeneous polynomial $f \in \mathbb{Z}[x_0, \dots, x_n]_d = \mathrm{Sym}^d(\mathbb{Z}^{\{0,\dots,n\}})$, we can consider the subfunctor $\mathrm{D}(f)$ of $\mathbb{P}^n$ consisting of the quotient lines $R^{\{0,\dots,n\}} \twoheadrightarrow L$ such that $L^{\otimes d}$ is generated by the image of f, or equivalently such that the composite

$$R \xrightarrow{f} \mathrm{Sym}^d_R(R^{\{0,\dots,n\}}) \twoheadrightarrow L^{\otimes d}$$

is an isomorphism. This is similar to the nonvanishing locus of a function, except that f is not quite a function on $\mathbb{P}^n$ (as we will see later, it is a section of the line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$). For example, $\mathrm{D}(x_i)$ can be identified with the affine n-space $\mathbb{A}^{\{0,\dots,\hat{\imath},\dots,n\}}$. The goal of this section is to show that $\mathrm{D}(f) \subset \mathbb{P}^n$ is an open subfunctor, which is moreover relatively affine, in the sense that its preimage in any affine scheme is affine.

We start with some generalities. Let P be a property of modules that is preserved by base change. For any ring A and A-module M, we can then consider the locus "where M has property P", which refers to the following subfunctor of $\mathrm{Spec}(A)$:

$$R \mapsto \{\varphi\colon A \to R \mid \text{the } R\text{-module } \varphi^*(M) \text{ has property } P\}.$$

For example, M has a zero locus, a finite presentation locus, a flatness locus, etc. Similarly, properties of linear maps, algebras, etc., that are preserved by base change have associated loci.

Definition 3.44 (Loci associated with a linear map)**.** Let A be a ring and let $f\colon M \to N$ be a map of A-modules. We define the following subfunctors of $\mathrm{Spec}(A)$:

- The *vanishing locus* $\mathrm{V}(f)$ of f is the locus where f is zero;
- The *epimorphism locus* $\mathrm{Epi}(f)$ of f is the locus where f is surjective;
- The *monomorphism locus* $\mathrm{Mono}(f)$ of f is the locus where f is *universally* injective;[4]
- The *isomorphism locus* $\mathrm{Iso}(f)$ of f is the locus where f is bijective.

If N is a line (resp. if M is a line), the epimorphism locus (resp. the monomorphism locus) of f is also called the *nonvanishing locus* of f and denoted by $\mathrm{D}(f)$.

Remark 3.45. Definition 2.55 for $X = \mathrm{Spec}(A)$ can be viewed as a special case of Definition 3.44: if $F \subset A$ is a subset and $f\colon A^{(F)} \to A$ is the induced linear map, then $\mathrm{V}(F) = \mathrm{V}(f)$ and $\mathrm{D}(F) = \mathrm{D}(f)$.

Remark 3.46. Let $f\colon V \to W$ be a map of vector spaces. By Proposition 3.14, we have

$$\mathrm{Mono}(f) = \mathrm{Epi}(f^\vee) \quad \text{and} \quad \mathrm{Iso}(f) = \mathrm{Epi}(f) \cap \mathrm{Epi}(f^\vee).$$

If moreover V and W have the same rank (e.g., are both lines), Corollary 3.17 implies that

$$\mathrm{Iso}(f) = \mathrm{Epi}(f) = \mathrm{Mono}(f).$$

Proposition 3.47. *Let A be a ring and let $f\colon M \to N$ be a map of A-modules.*

(i) *If N is a vector space, then* $\mathrm{V}(f) \subset \mathrm{Spec}(A)$ *is a closed subfunctor.*
(ii) *If N is a vector space, then* $\mathrm{Epi}(f) \subset \mathrm{Spec}(A)$ *is an open subfunctor.*
(iii) *If M and N are vector spaces, then* $\mathrm{Mono}(f) \subset \mathrm{Spec}(A)$ *is an open subfunctor.*
(iv) *If M and N are vector spaces, then* $\mathrm{Iso}(f) \subset \mathrm{Spec}(A)$ *is an open subfunctor.*

We now study more closely the nonvanishing locus of a map of lines $s\colon L' \to L$ over R. Since L' is invertible, we can tensor s with the inverse of L' without changing its nonvanishing locus, so it suffices to consider the case $L' = R$. We can then identify the R-linear map $s\colon R \to L$ with an element $s \in L$. In the special case $L = R$, the theory of localization of rings tells us that the nonvanishing locus of any $s \in R$ is isomorphic to $\mathrm{Spec}(R_s)$ (see Proposition 2.62(ii)). This theory can be generalized to nontrivial lines as follows.

Definition 3.48 (Periodic module)**.** Let R be a ring, L a line over R, and $s \in L$. An R-module M is *s-periodic* if the map

$$\mathrm{id}_M \otimes s\colon M \otimes_R R \to M \otimes_R L$$

is an isomorphism.

Proposition 3.49. *Let R be a ring, L a line over R, and $s \in L$.*

(i) *There exists an initial s-periodic R-algebra R_s.*
(ii) *The forgetful functor* $\mathrm{Mod}_{R_s} \to \mathrm{Mod}_R$ *identifies* Mod_{R_s} *with the full subcategory of s-periodic R-modules in* Mod_R*. Hence, the inclusion of this subcategory has a left adjoint $M \mapsto M_s$ given by $M_s = M \otimes_R R_s$.*
(iii) *The R-module M_s can be computed as the colimit of the sequence*

$$M \xrightarrow{s} M \otimes_R L \xrightarrow{s} M \otimes_R L^{\otimes 2} \xrightarrow{s} \cdots .$$

Corollary 3.50. *Let R be a ring, L a line over R, and $s \in L$. Then there is a canonical isomorphism* $\mathrm{Spec}(R_s) \xrightarrow{\sim} \mathrm{D}(s) \subset \mathrm{Spec}(R)$*. In particular,* $\mathrm{D}(s)$ *is an affine scheme.*

Remark 3.51 (Improved Zariski descent)**.** All the Zariski descent results from §2.8 can be generalized by replacing the family $(f_i)_{i\in I}$ generating the unit ideal in R with a family $(s_i)_{i\in I}$ generating some line L over R. For example, the generalized version of Corollary 2.72 says that, for any affine scheme X, the diagram

$$X(R) \to \prod_{i\in I} X(R_{s_i}) \rightrightarrows \prod_{i,j\in I} X(R_{s_is_j})$$

is an equalizer diagram (where $s_is_j \in L^{\otimes 2}$). For each result, we can either repeat the proof with no essential changes, or reduce the new statement to the old one using that $\mathrm{D}(s) = \mathrm{D}(F_s)$, where $F_s \subset R$ is the image by $s^\vee\colon L^\vee \to R$ of a generating family of $L^\vee$.

[4] We need to use universal injectivity to define the monomorphism locus, as injectivity is not preserved by base change.

Example 3.52 (Projective completions of affine spaces)**.** Let k be a ring and M a k-module. Recall the algebraic k-functors $\mathbb{A}(M)$ and $\mathbb{P}(M)$ from Examples 2.37 and 3.42. There is a canonical embedding

$$\mathbb{A}(M) \hookrightarrow \mathbb{P}(M \oplus k), \quad (a\colon M \otimes_k R \to R) \mapsto \big((a, \mathrm{id}_R)\colon (M \otimes_k R) \oplus R \twoheadrightarrow R\big).$$

We claim that this is an open immersion. Indeed, given any map $\mathrm{Spec}(R) \to \mathbb{P}(M \oplus k)$ classifying a quotient line $(M \otimes_k R) \oplus R \twoheadrightarrow L$, the preimage of $\mathbb{A}(M)$ in $\mathrm{Spec}(R)$ is exactly the isomorphism locus of the composite

$$R \hookrightarrow (M \otimes_k R) \oplus R \twoheadrightarrow L,$$

which is open in $\mathrm{Spec}(R)$ by Proposition 3.47(iv) (and also affine by Corollary 3.50) Taking $M = k^{(I)}$, we get a canonical open immersion

$$\mathbb{A}^I \hookrightarrow \mathbb{P}^{I \sqcup \{0\}},$$

identifying $\mathbb{A}^I$ with $\mathrm{D}(x_0) \subset \mathbb{P}^{I \sqcup \{0\}}$. For $I = \{1, \dots, n\}$, this is an open immersion $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$.

3.7. **The functor** Proj**.** In this section, we define $\mathrm{Proj}(A)$ for any $\mathbb{N}$-graded ring A. If A is finitely generated as an A_0-algebra, which is usually the case in practice, we will see that $\mathrm{Proj}(A)$ reduces to Construction 3.40. Thus, the increase in generality is minimal. Instead, the main result of this section is that Construction 3.40 satisfies

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)}).$$

This isomorphism gives rise to the *Veronese embedding*, which is a central feature of projective geometry.

Definition 3.53 (Eventually surjective map)**.** A map of $\mathbb{N}$-graded rings $A \to B$ is called *eventually surjective* if for every $d \geq 1$ and every $b \in B_d$ there exists $n \geq 1$ such that $A_{nd} \otimes_{A_0} B_0 \to B_{nd}$ hits b^n. We denote by $\mathrm{CAlg}^{\mathbb{N},\mathrm{es}}$ the category of $\mathbb{N}$-graded rings and eventually surjective maps.

Recall from Notation 3.32 that $A^{(d)} = \bigoplus_{n \in \mathbb{N}} A_{nd}$ and $A_{(f)} = (A_f)_0$.

Construction 3.54. We construct a functor

$$\mathrm{Proj}\colon (\mathrm{CAlg}^{\mathbb{N},\mathrm{es}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$$

with a natural transformation $\mathrm{Proj}(-) \to \mathrm{Spec}((-)_0)$. If A is an $\mathbb{N}$-graded k-algebra, we can thus view $\mathrm{Proj}(A)$ as an algebraic k-functor, and we will see in Theorem 3.59(ii) that it recovers Construction 3.40 when A is generated by A_1.

Let A be an $\mathbb{N}$-graded ring. We first define

$$\mathrm{Proj}_1(A) \in \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/\,\mathrm{Spec}(A_0)} \simeq \mathrm{Fun}(\mathrm{CAlg}_{A_0}, \mathrm{Set})$$

as follows. Given an A_0-algebra R, $\mathrm{Proj}_1(A)(R)$ is the set of pairs (L, φ), where L is a quotient line of the R-module $A_1 \otimes_{A_0} R$ and $\varphi\colon A \otimes_{A_0} R \to \mathrm{Sym}_R(L)$ is a map of graded R-algebras extending the quotient map $A_1 \otimes_{A_0} R \twoheadrightarrow L$. Note that φ is automatically surjective, so that we can alternatively describe $\mathrm{Proj}_1(A)(R)$ as the set of quotient $\mathbb{N}$-graded R-algebras of $A \otimes_{A_0} R$ that are isomorphic to symmetric algebras of lines.

For $d \in \mathbb{N}$, we define $\mathrm{Proj}_d(A) = \mathrm{Proj}_1(A^{(d)})$. For any $n \in \mathbb{N}$, there is then a canonical map

$$\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A), \quad (L, \varphi) \mapsto (L^{\otimes n}, \varphi^{(n)}),$$

where $L^{\otimes n}$ is a quotient of $A_{nd} \otimes_{A_0} R$ via φ_n. This defines a functor

$$\mathbb{N}^{\mathrm{div}} \to \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}), \quad d \mapsto \mathrm{Proj}_d(A),$$

where $\mathbb{N}^{\mathrm{div}}$ is the poset of natural numbers under divisibility. We then set

$$\mathrm{Proj}(A) = \operatorname*{colim}_{d \in \mathbb{N}^{\mathrm{div}}_{>0}} \mathrm{Proj}_d(A).$$

Since 0 is the final object of $\mathbb{N}^{\mathrm{div}}$, this comes with a canonical map

$$\mathrm{Proj}(A) \to \mathrm{Proj}_0(A) = \mathrm{Proj}_1(A^{(0)}) = \mathrm{Proj}_1(A_0[x]) = \mathbb{P}^0_{A_0} = \mathrm{Spec}(A_0).$$

Let $\alpha\colon A \to B$ be a map of $\mathbb{N}$-graded rings and let (L, φ) be an R-point of $\mathrm{Proj}_d(B)$ for some $d \geq 1$. Then there exist $b_1, \dots, b_r \in B_d$ whose images by φ_d generate L. If for some $n \geq 1$, the powers $b_1^n, \dots, b_r^n$ are in the image of $\alpha_{nd}\colon A_{nd} \otimes_{A_0} B_0 \to B_{nd}$, then $(L^{\otimes n}, \varphi^{(n)} \circ \alpha^{(nd)})$ is an R-point of $\mathrm{Proj}_{nd}(A)$. Thus, if α is eventually surjective, precomposition with α defines a map $\mathrm{Proj}(\alpha)\colon \mathrm{Proj}(B) \to \mathrm{Proj}(A)$.

Remark 3.55. For every $d \geq 1$, the inclusion $d\mathbb{N}^{\mathrm{div}}_{>0} \subset \mathbb{N}^{\mathrm{div}}_{>0}$ is cofinal and hence induces an isomorphism

$$\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)}).$$

Remark 3.56 (Degree d maps)**.** We can enlarge the domain of definition of Proj as follows. If A and B are $\mathbb{N}$-graded rings and $d \in \mathbb{N}$, a *degree d map* from A to B is a graded ring map $A \to B^{(d)}$. If it is eventually surjective and $d \geq 1$, then, by Remark 3.55, it induces

$$\mathrm{Proj}(B) \simeq \mathrm{Proj}(B^{(d)}) \to \mathrm{Proj}(A).$$

This also works if $d = 0$: $\mathrm{Proj}(B) \to \mathrm{Spec}(B_0) = \mathrm{Proj}(B^{(0)}) \to \mathrm{Proj}(A)$. We can compose a degree d map with a degree e map to obtain a degree $d + e$ map. Thus, there is an enlargement of the category $\mathrm{CAlg}^{\mathbb{N},\mathrm{es}}$ with the same objects and in which the set of maps from A to B is $\coprod_{d\in\mathbb{N}} \mathrm{Map}_{\mathrm{CAlg}^{\mathbb{N},\mathrm{es}}}(A, B^{(d)})$, and the functor Proj extends to this category.

Proposition 3.57 (Zariski descent for Proj)**.** *Let A be an $\mathbb{N}$-graded ring, let $d \in \mathbb{N}$, and let X be either $\mathrm{Proj}_d(A)$ of $\mathrm{Proj}(A)$. For every ring R, every line L over R, and every generating family $(s_i)_{i\in I}$ of L, the diagram*

$$X(R) \to \prod_{i\in I} X(R_{s_i}) \rightrightarrows \prod_{i,j\in I} X(R_{s_i s_j})$$

is an equalizer.

Lemma 3.58. *Let A be an $\mathbb{N}$-graded ring, let $d \geq 1$, and let $f \in A_d$. Let $\mathrm{D}(f) \subset \mathrm{Proj}_d(A)$ be the subfunctor of pairs (L, φ) such that $\varphi(f)$ generates L (which is open by Proposition 3.47).*

(i) *There is a canonical isomorphism $\mathrm{D}(f) \simeq \mathrm{Spec}(A_{(f)})$.*

(ii) *For every $n \geq 1$, the map $\mathrm{Proj}_d(A) \to \mathrm{Proj}_{nd}(A)$ induces an isomorphism $\mathrm{D}(f) \xrightarrow{\sim} \mathrm{D}(f^n)$. Moreover, the square*

$$\begin{array}{ccc} \mathrm{D}(f) & \xrightarrow{\sim} & \mathrm{D}(f^n) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Proj}_d(A) & \longrightarrow & \mathrm{Proj}_{nd}(A) \end{array}$$

is cartesian.

(iii) *Let $f_1, \dots, f_k \in A$ be homogeneous elements of positive degrees. Under the isomorphisms of* (ii)*, we have $\mathrm{D}(f_1 \cdots f_k) = \mathrm{D}(f_1) \cap \dots \cap \mathrm{D}(f_k)$ inside $\mathrm{Proj}_d(A)$ for all sufficiently divisible d.*

Theorem 3.59 (Structure of Proj)**.** *Let A be an $\mathbb{N}$-graded ring and let $d \geq 1$.*

(i) *The map $\mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ is an open immersion.*

(ii) *Suppose that A is generated as an A_0-algebra by homogeneous elements whose degrees divide d. Then the map $\mathrm{Proj}_d(A) \to \mathrm{Proj}(A)$ is an isomorphism. In particular, if A is generated by $A_{\leq 1}$, then $\mathrm{Proj}_1(A) \xrightarrow{\sim} \mathrm{Proj}(A)$.*

Corollary 3.60. *Let A be a finitely generated $\mathbb{N}$-graded k-algebra such that $k \to A_0$ is surjective. Then $\mathrm{Proj}(A)$ is a projective k-scheme.*

Remark 3.61. The functor Proj functor extends the functor Spec as follows: there is a commutative triangle (up to natural isomorphism)

$$\begin{array}{ccc} \mathrm{CAlg} & \xrightarrow{(-)[x]} & \mathrm{CAlg}^{\mathbb{N},\mathrm{es}} \\ & {\scriptstyle\mathrm{Spec}} \searrow & \big\downarrow {\scriptstyle\mathrm{Proj}} \\ & & \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}), \end{array}$$

where the horizontal functor sends R to the $\mathbb{N}$-graded ring $R[x]$. This follows from Theorem 3.59(ii) as $\mathrm{Proj}(R[x]) \simeq \mathrm{Proj}_1(R[x]) = \mathbb{P}^0_R = \mathrm{Spec}(R)$.

Definition 3.62 (Irrelevant ideal)**.** Let A be an $\mathbb{N}$-graded ring. The *irrelevant ideal* of A is the homogeneous ideal $A_+ = \bigoplus_{d\geq 1} A_d$.

Remark 3.63 (Spec vs. Proj)**.** The relation between $\mathbb{A}^I$ and $\mathbb{P}^I$ from §3.3 extends to a relation between $\mathrm{Spec}(A)$ and $\mathrm{Proj}(A)$. Namely, let $\mathrm{D}(A_+) \subset \mathrm{Spec}(A)$ be the nonvanishing locus of the irrelevant ideal. Then there is a zigzag

$$\mathrm{Spec}(A) \hookleftarrow \mathrm{D}(A_+) \to \mathrm{Proj}(A)$$

over $\mathrm{Spec}(A_0)$, natural in $A \in \mathrm{CAlg}^{\mathbb{N},\mathrm{es}}$, where the second map is defined as follows. Since any family generating the unit ideal contains a finite generating subfamily, we have

$$\mathrm{D}(A_+) = \operatorname*{colim}_{d\in\mathbb{N}^{\mathrm{div}}_{>0}} \mathrm{D}(A_d).$$

The map $\mathrm{D}(A_+) \to \mathrm{Proj}(A)$ is then the colimit of the maps

$$\mathrm{D}(A_d) \to \mathrm{Proj}_d(A), \quad (\varphi\colon A \to R) \mapsto \left(\textstyle\bigoplus_{n\in\mathbb{N}} \varphi|_{A_{nd}} \colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(R)\right).$$

There is a canonical action of the affine group scheme $\mathbb{G}_\mathrm{m}$ on $\mathrm{Spec}(A)$, which restricts to each $\mathrm{D}(A_d)$: for any $\varphi\colon A \to R$ and $\lambda \in R^\times$, we define $\lambda\varphi\colon A \to R$ by

$$(\lambda\varphi)(a) = \lambda^d \varphi(a) \quad \text{for all } a \in A_d.$$

Each map $\mathrm{D}(A_d) \to \mathrm{Proj}_d(A)$ is then $\mathbb{G}_\mathrm{m}$-invariant, and we have the following generalization of Proposition 3.24: the induced map $\mathrm{D}(A_1)/\mathbb{G}_\mathrm{m} \to \mathrm{Proj}_1(A)$ is a monomorphism, whose image consists exactly of the trivial quotient lines; in particular, it is a bijection on local rings and principal ideal domains. This does not hold for arbitrary $d \in \mathbb{N}$, but it will turn out that $\mathrm{Proj}_d(A)$ is still the quotient $\mathrm{D}(A_d)/\mathbb{G}_\mathrm{m}$ in the category of *schemes*.

Definition 3.64 (Vanishing and nonvanishing loci)**.** Let A be an $\mathbb{N}$-graded ring and let $F \subset A$ be a homogeneous subset.

(i) The *vanishing locus* of F is the subfunctor $\mathrm{V}(F) \subset \mathrm{Proj}(A)$, which is the union over $d \in \mathbb{N}^{\mathrm{div}}_{>0}$ of the subfunctors $\mathrm{V}_d(F) \subset \mathrm{Proj}_d(A)$ given by

$$\mathrm{V}_d(F)(R) = \{\varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \mid \varphi(f) = 0 \text{ for all } f \in (F)^{(d)}\}.$$

(ii) The *nonvanishing locus* of F is the subfunctor $\mathrm{D}(F) \subset \mathrm{Proj}(A)$, which is the union over $d \in \mathbb{N}^{\mathrm{div}}_{>0}$ of the subfunctors $\mathrm{D}_d(F) \subset \mathrm{Proj}_d(A)$ given by

$$\mathrm{D}_d(F)(R) = \{\varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \mid \varphi((F)_{nd}) \text{ generates } L^{\otimes n} \text{ for some } n \in \mathbb{N}\}.$$

Remark 3.65. One can easily check that V and D satisfy exactly the same formal properties as in the affine case (Proposition 2.60). Note also that $\mathrm{V}(F)$ depends only on (F) and $\mathrm{D}(F)$ only on $\sqrt{(F)}$. If A is generated by $A_{\leq 1}$, we have the following simpler descriptions as subfunctors of $\mathrm{Proj}_1(A) \simeq \mathrm{Proj}(A)$:

$$\mathrm{V}(F)(R) = \{\varphi\colon A \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \mid \varphi(f) = 0 \text{ for all } f \in F\},$$

$$\mathrm{D}(F)(R) = \{\varphi\colon A \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L) \mid (\varphi(F)) \text{ contains } L^{\otimes n} \text{ for some } n \in \mathbb{N}\}.$$

Remark 3.66. Let A be an $\mathbb{N}$-graded ring.

(i) For any homogeneous subset $F \subset A$, there is a canonical isomorphism $\mathrm{Proj}(A/(F)) \xrightarrow{\sim} \mathrm{V}(F)$.

(ii) Let $f \in A_d$ with $d \geq 1$. Then the nonvanishing locus $\mathrm{D}(f)$ of Definition 3.64 is contained in $\mathrm{Proj}_d(A)$ and matches the one from Lemma 3.58, by part (ii) of the lemma. By part (i) of the lemma, there is a canonical isomorphism $\mathrm{Spec}(A_{(f)}) \xrightarrow{\sim} \mathrm{D}(f)$.

(iii) If $f \in A_0$, there is a canonical isomorphism $\mathrm{Proj}(A_f) \xrightarrow{\sim} \mathrm{D}(f)$.

Proposition 3.67. *Let A be an $\mathbb{N}$-graded ring and let $F \subset A$ be a homogeneous subset. Then:*

(i) $\mathrm{V}(F)$ *is a closed subfunctor of* $\mathrm{Proj}(A)$*.*

(ii) $\mathrm{D}(F)$ *is an open subfunctor of* $\mathrm{Proj}(A)$*.*

Example 3.68 (The Veronese embedding)**.** Let k be a ring, M a k-module, and $d \in \mathbb{N}$. The *Veronese embedding of degree d* is the map of algebraic k-functors

$$\rho_d\colon \mathbb{P}(M) \to \mathbb{P}(\mathrm{Sym}^d_k(M))$$

sending a quotient line $M \otimes_k R \twoheadrightarrow L$ to its dth symmetric power $\mathrm{Sym}^d_k(M) \otimes_k R \twoheadrightarrow \mathrm{Sym}^d_R(L) = L^{\otimes d}$. If $M = k^{(I)}$, the Veronese embedding of degree d is a map

$$\rho_d\colon \mathbb{P}^I_k \to \mathbb{P}^{\mathrm{Sym}^d(I)}_k,$$

where $\mathrm{Sym}^d(I)$ is the quotient of I^d by the action of the symmetric group Σ_d. If $M = k^{n+1}$, this becomes the map

$$\rho_d\colon \mathbb{P}^n_k \to \mathbb{P}^{\binom{n+d}{n}-1}_k,$$

which is given in coordinates by the formula

$$[a_0 : \dots : a_n] \mapsto [\text{all degree } d \text{ monomials in the } a_i\text{'s}].$$

We claim that ρ_d is a closed immersion if $d \geq 1$. Indeed, ρ_d is Proj of a surjective degree d map of graded rings. Explicitly, it is the composite

$$\mathrm{Proj}_1(\mathrm{Sym}_k(M)) \to \mathrm{Proj}_d(\mathrm{Sym}_k(M)) \to \mathrm{Proj}_1(\mathrm{Sym}_k(\mathrm{Sym}^d_k(M))),$$

where the first map sends (L, φ) to $(L^{\otimes d}, \varphi^{(d)})$ and the second map is Proj_1 of the surjective map

$$\mathrm{Sym}_k(\mathrm{Sym}^d_k(M)) \twoheadrightarrow \mathrm{Sym}_k(M)^{(d)}.$$

If $d \geq 1$, the first map is an isomorphism by Theorem 3.59(ii), and the second map is a closed immersion by Proposition 3.67(i).

Analogously to quasi-affine scheme, we define quasi-projective schemes as nonvanishing loci in projective schemes:

Definition 3.69 (Quasi-projective scheme)**.** Let k be a ring. An algebraic k-functor X is a *quasi-projective k-scheme* if there exists an $\mathbb{N}$-graded k-algebra A, generated by a finite subset of A_1, and a *finite* homogeneous subset $F \subset A$ such that $X \simeq \mathrm{D}(F) \subset \mathrm{Proj}(A)$. We denote by $\mathrm{QProj}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ the full subcategory of quasi-projective k-schemes.

Note that we could have assumed $F \subset A_1$ without changing the definition: we can assume $F \subset A_+$ by multiplying F by a finite generating subset of A_1, and if $d \geq 1$ is divisible by all degrees in F, we can then replace A by $A^{(d)}$ and each element of F by its power in A_d. We will see in §3.9 that every quasi-affine k-scheme of finite type is quasi-projective. Thus, under this finiteness assumption, quasi-projective k-schemes subsume all types of k-schemes discussed so far.

3.8. **Saturation.** The goal of this section is to classify closed and open subfunctors of $\mathrm{Proj}(A)$. This is more subtle than the analogous result for $\mathrm{Spec}(A)$ (Proposition 2.66), since there can be different homogeneous ideals in A with the same vanishing locus in $\mathrm{Proj}(A)$:

Example 3.70. The homogeneous ideals (x) and (x^2, xy) in $\mathbb{Z}[x, y]$ have the same vanishing locus in $\mathbb{P}^1 = \mathrm{Proj}(\mathbb{Z}[x, y])$. Since $(x^2, xy) \subset (x)$, we have $\mathrm{V}(x) \subset \mathrm{V}(x^2, xy)$. For the converse, consider an arbitrary quotient line $(a, b)\colon R^2 \twoheadrightarrow L$ in $\mathbb{P}^1(R)$ such that $a^2 = 0$ and $ab = 0$ in $L^{\otimes 2}$. Since $\mathrm{Sym}^2_R(R^2) \to L^{\otimes 2}$ is surjective, $L^{\otimes 2}$ is generated by (a^2, ab, b^2), hence by b^2 alone. Since L is invertible, this implies that b generates L. From $ab = 0$ we then deduce that $a = 0$, as desired.

Definition 3.71 (Saturation)**.** Let A be an $\mathbb{N}$-graded ring and let $H \subset A$ be a homogeneous ideal. The *saturation* of H is the homogeneous ideal

$$H^{\mathrm{sat}} = \{x \in A \mid \text{for all } f \in A_+ \text{ there exists } n \in \mathbb{N} \text{ such that } f^n x \in H\}.$$

We say that H is *saturated* if $H = H^{\mathrm{sat}}$.

Remark 3.72.

(i) Saturated ideals are determined by their "tail": if $H \subset A$ is a homogeneous ideal and $D \subset \mathbb{N}$ is an infinite subset, then $H^{\mathrm{sat}} = \big(\bigoplus_{d \in D} H_d\big)^{\mathrm{sat}}$.
(ii) If A is generated by $A_{\leq 1}$, we can replace A_+ by A_1 in the definition of H^{sat}.
(iii) If $H \subset A$ is a radical homogeneous ideal, then H^{sat} is also radical.
(iv) A homogeneous prime ideal $\mathfrak{p} \subset A$ is saturated if and only if $\mathfrak{p}$ does not contain A_+.

Example 3.73.

(i) In $\mathbb{Z}[x, y]$, we have $(x^2, xy)^{\mathrm{sat}} = (x)$.
(ii) In any polynomial ring $k[x_i \mid i \in I]$, (0) is saturated.

We briefly consider a more general context for this definition:

Definition 3.74 (I-nilpotent, I-local module)**.** Let R be a ring, $I \subset R$ a subset, and M an R-module. An element $x \in M$ is called *I-nilpotent* if, for every $f \in I$, there exists $n \in \mathbb{N}$ such that $f^n x = 0$. The I-nilpotent elements of M form a submodule $\Gamma_I M \subset M$. The module M is called:

- *I-nilpotent* if $\Gamma_I M = M$;
- *I-local* if every R-linear map $h\colon P \to Q$ with I-nilpotent kernel and cokernel induces a bijection

$$h^*\colon \mathrm{Map}(Q, M) \xrightarrow{\sim} \mathrm{Map}(P, M).$$

We denote the corresponding full subcategories of Mod_R by $\mathrm{Mod}_R^{I\text{-nil}}$ and $\mathrm{Mod}_R^{I\text{-loc}}$.

Note that these conditions depend only on the radical ideal generated by I. The categories $\mathrm{Mod}_R^{I\text{-nil}}$ and $\mathrm{Mod}_R^{I\text{-loc}}$ are abelian and fit in a "short exact sequence" of Grothendieck abelian categories

$$\mathrm{Mod}_R^{I\text{-nil}} \underset{\Gamma_I}{\rightleftarrows} \mathrm{Mod}_R \overset{\mathrm{L}_I}{\rightleftarrows} \mathrm{Mod}_R^{I\text{-loc}},$$

where the left adjoint functors are exact. In particular, an R-module M is I-nilpotent if and only if $\mathrm{L}_I M = 0$, and $\Gamma_I M$ is the kernel of the unit map $M \to \mathrm{L}_I M$.

Remark 3.75. An R-module M is I-nilpotent if and only if $M_f = 0$ for all $f \in I$. Hence, an R-linear map $M \to N$ induces an isomorphism $\mathrm{L}_I M \xrightarrow{\sim} \mathrm{L}_I N$ if and only if it induces isomorphisms $M_f \xrightarrow{\sim} N_f$ for all $f \in I$. In particular, for any $f \in R$, "$\{f\}$-local" is synonymous with "f-periodic", and $\mathrm{L}_{\{f\}} M = M_f$.

Remark 3.76. We will give later a geometric interpretation of $\mathrm{Mod}_R^{I\text{-loc}}$ when I is finite (Theorem 4.49): it can be identified with the category of quasi-coherent modules on the open subscheme $\mathrm{D}(I) \subset \mathrm{Spec}(R)$. In this case, $\mathrm{L}_I M$ can be computed explicitly as the equalizer

$$\mathrm{L}_I M \to \prod_{f \in I} M_f \rightrightarrows \prod_{f,g \in I} M_{fg},$$

and we have already seen in Corollary 2.74 that $\mathcal{O}(\mathrm{D}(I)) = \mathrm{L}_I(R)$.

Warning 3.77. For a subset $I \subset R$, the construction $\mathrm{L}_I M$ should not be confused with the construction $M[I^{-1}]$, which are both called "localization". They coincide when I has a single element but generalize this case in different directions: for any $f \in I$, there are canonical maps

$$\mathrm{L}_I M \to \mathrm{L}_{\{f\}} M = M_f \to M[I^{-1}].$$

The geometric interpretation of these constructions is the following. If I is finite, then $\mathrm{L}_I R$ is the ring of functions on $\mathrm{D}(I)$, which is the union $\bigvee_{f \in I} \mathrm{D}(f)$ in the poset of open subfunctors of $\mathrm{Spec}(R)$ and is usually not affine. On the other hand, the intersection $\bigcap_{f \in I} \mathrm{D}(f)$ is always affine (but usually not open in $\mathrm{Spec}(R)$, unless I is finite) and isomorphic to $\mathrm{Spec}(R[I^{-1}])$.

Example 3.78.

(i) Let L be a line over R and let $s\colon R \to L$. Then an R-module is s-periodic if and only if it is $\mathrm{im}(s^\vee)$-local.

(ii) Let A be an $\mathbb{N}$-graded ring. The saturation of a homogeneous ideal $H \subset A$ is the kernel of $A \to \mathrm{L}_{A_+}(A/H)$. In particular, $(0)^{\mathrm{sat}} = \Gamma_{A_+} A$.

Proposition 3.79 (Conservativity of Proj)**.** *Let A and B be $\mathbb{N}$-graded rings and let $A \to B$ be an eventually surjective map. The following conditions are equivalent:*

(i) $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ *is an isomorphism.*

(ii) *For all $d \geq 1$, $\mathrm{L}_{A_d} A^{(d)} \to \mathrm{L}_{A_d} B^{(d)}$ is an isomorphism.*

If $A \to B$ is surjective, these are further equivalent to:

(iii) *For all $d \geq 1$, $\ker(A \to B)^{(d)}$ is A_d-nilpotent.*

If A is generated by $A_{\leq 1}$, it suffices to consider $d = 1$ in (ii) *and* (iii).

Corollary 3.80 (Functorial projective Nullstellensatz)**.** *Sending a homogeneous subset $F \subset k[x_i \mid i \in I]$ to its vanishing locus $\mathrm{V}(F) \subset \mathbb{P}^I_k$ induces an order-reversing bijection*

$$\mathrm{V}\colon \{\text{saturated homogeneous ideals in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\text{vanishing loci in } \mathbb{P}^I_k\}.$$

For simplicity, we state the next result for $\mathbb{N}$-graded rings that are generated in degrees ≤ 1.

Proposition 3.81 (Classification of closed and open subfunctors of projective schemes)**.** *Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$.*

(i) *The construction $F \mapsto \mathrm{V}(F)$ induces an order-reversing injection*

$$\{\textit{saturated homogeneous ideals in } A\} \hookrightarrow \{\textit{closed subfunctors of } \mathrm{Proj}(A)\},$$

which is bijective if A is finitely generated as an A_0-algebra.

(ii) *The construction $F \mapsto \mathrm{D}(F)$ induces an order-preserving bijection*

$$\{\textit{saturated radical homogeneous ideals in } A\} \xrightarrow{\sim} \{\textit{open subfunctors of } \mathrm{Proj}(A)\}.$$

Remark 3.82. Proposition 3.81 generalizes to all $\mathbb{N}$-graded rings A as follows: (ii) remains unchanged, but in (i) we must replace "saturated" by the following stronger condition on an ideal H: H contains every homogeneous element $x \in A$ that satisfies $(x)_d \subset \ker(A \to \mathrm{L}_{A_d}(A/H))$ for all $d \geq 1$. This is arguably the correct definition of "saturated" in general, but it agrees with Definition 3.71 when A is generated by $A_{\leq 1}$ or when H is radical.

Remark 3.83. For any $\mathbb{N}$-graded ring A, the construction $F \mapsto F \cap A_+$ defines a bijection

$$\{\text{saturated radical homogeneous ideals in } A\} \xrightarrow{\sim} \{\text{homogeneous ideals } H \subset A_+ \text{ with } \sqrt{H} \cap A_+ = H\}$$

with inverse $H \mapsto H^{\mathrm{sat}}$. This gives an alternative formulation of Proposition 3.81(ii).

It follows from Proposition 3.81(i) that every closed subfunctor of a projective k-scheme is again a projective k-scheme. We also have the following analogue of Proposition 2.80:

Proposition 3.84. *Let X be a quasi-projective k-scheme and let $Z \hookrightarrow X$ be a closed immersion. Then Z is a quasi-projective k-scheme.*

3.9. **Projective closure.** Recall from Example 3.52 that any affine space $\mathbb{A}^I$ admits a canonical open embedding $\mathbb{A}^I \hookrightarrow \mathbb{P}^{I\sqcup\{0\}}$, which identifies $\mathbb{A}^I$ with the nonvanishing locus of x_0 in $\mathbb{P}^{I\sqcup\{0\}}$. The goal of this section is to compare vanishing and nonvanishing loci in $\mathbb{A}^I$ and $\mathbb{P}^{I\sqcup\{0\}}$.

Proposition 3.85. *Let I be a set not containing 0, let $F \subset k[x_0, x_i \,|\, i \in I]$ be a homogeneous subset, and let $F_0 \subset k[x_i \mid i \in I]$ be obtained from F by setting $x_0 = 1$. Consider the vanishing loci $\mathrm{V}(F) \subset \mathbb{P}_k^{I\sqcup\{0\}}$ and $\mathrm{V}(F_0) \subset \mathbb{A}_k^I$ and the nonvanishing loci $\mathrm{D}(F) \subset \mathbb{P}_k^{I\sqcup\{0\}}$ and $\mathrm{D}(F_0) \subset \mathbb{A}_k^I$. Then*

$$\mathrm{V}(F) \cap \mathbb{A}_k^I = \mathrm{V}(F_0) \quad \textit{and} \quad \mathrm{D}(F) \cap \mathbb{A}_k^I = \mathrm{D}(F_0).$$

Next, we want to describe the projective closure of affine vanishing loci.

Definition 3.86 (Closure of subfunctor)**.** Let X be an algebraic functor and $Y \subset X$ a subfunctor. The *closure* of Y in X is the smallest closed subfunctor of X containing Y. This is well-defined as arbitrary intersections of closed subfunctors are closed (by Proposition 2.60(i)).

Definition 3.87 (Homogenization of polynomials)**.** Let k be a ring and I a set (not containing 0).

(i) Let $f \in k[x_i \mid i \in I]$. The *homogenization* of f is the homogeneous polynomial

$$f^{\mathrm{h}} = x_0^d f\big(\big(\tfrac{x_i}{x_0}\big)_{i\in I}\big) \in k[x_0, x_i \mid i \in I],$$

where d is the maximal degree of a monomial with nonzero coefficient in f, or equivalently the smallest integer for which the above expression is a polynomial.

(ii) Let $F \subset k[x_i \mid i \in I]$ be an ideal. The *homogenization* of F is the homogeneous ideal

$$F^{\mathrm{h}} = (f^{\mathrm{h}} \mid f \in F) \subset k[x_0, x_i \mid i \in I].$$

Remark 3.88.

(i) We can recover any polynomial or ideal from its homogenization by setting $x_0 = 1$. In fact, the homogeneous polynomials in F^{h} are exactly those whose evaluation at $x_0 = 1$ is in F. It follows that homogenization preserves radical and prime ideals.

(ii) We have $(f)^{\mathrm{h}} = (f^{\mathrm{h}})$, but this does not generalize to ideals with two or more generators. For example, if F is the ideal in $k[x_1, x_2]$ generated by $f = x_1 + x_2^2$ and $g = x_2$, then $F^{\mathrm{h}} = (x_1, x_2)$, but the ideal generated by $f^{\mathrm{h}} = x_0x_1 + x_2^2$ and $g^{\mathrm{h}} = x_2$ does not contain x_1.

Corollary 3.89 (Projective closure of vanishing loci)**.** *Let I be a finite set and $F \subset k[x_i \,|\, i \in I]$ an ideal with vanishing locus $\mathrm{V}(F) \subset \mathbb{A}_k^I$. Then $\mathrm{V}(F^{\mathrm{h}}) \subset \mathbb{P}_k^{I\sqcup\{0\}}$ is the closure of $\mathrm{V}(F)$.*

Corollary 3.90 (Quasi-affine schemes of finite type are quasi-projective)**.** *Let k be a ring, A a k-algebra of finite type, and $F \subset A$ a finite subset. Then the subfunctor $\mathrm{D}(F) \subset \mathrm{Spec}(A)$ is a quasi-projective k-scheme.*

Definition 3.91 (Finite scheme)**.** Let k be a ring. An algebraic k-functor X is a *finite k-scheme* if there exists a finite k-algebra A such that $X \simeq \mathrm{Spec}(A)$.

Proposition 3.92 (Finite schemes are projective)**.** *Let $X \subset \mathbb{A}_k^I$ be a closed subfunctor such that every function on X is integral over k, i.e., satisfies a monic polynomial equation over k. Then X is closed in $\mathbb{P}_k^{I\sqcup\{0\}}$. In particular, every finite k-scheme is projective.*

Remark 3.93. Conversely, if an affine k-scheme is a projective, then it is finite; we will prove this later using the notion of properness. This means that any closed subscheme of $\mathbb{A}_k^n$, if not finite, must "go to infinity" and not remain closed in $\mathbb{P}_k^n$. This is in stark contrast to the analogous situation in differential geometry, where there exist positive-dimensional closed submanifolds of $\mathbb{R}^n$ that are "away from infinity", i.e., remain closed in $\mathbb{RP}^n$.

3.10. **Examples of projective schemes.**

Example 3.94 (Weighted projective spaces)**.** Let I be a set and $w\colon I \to \mathbb{N}_{>0}$ a map. Then we we can equip the polynomial k-algebra $k[x_i \mid i \in I]$ with the $\mathbb{N}$-graded structure in which x_i is homogeneous of degree $w(i)$. The resulting algebraic k-functor $\mathbb{P}_k^w = \mathrm{Proj}(k[x_i \mid i \in I])$ is called the *weighted projective space* with weights w. It is a projective k-scheme if I is a finite, by Corollary 3.60.

Example 3.95 (The Segre embedding)**.** Let k be a ring and let M and N be k-modules. The *Segre embedding* is the map of algebraic k-functors

$$\varsigma\colon \mathbb{P}(M) \times \mathbb{P}(N) \to \mathbb{P}(M \otimes_k N)$$

that sends a pair of quotient lines $M \otimes_k R \twoheadrightarrow K$ and $N \otimes_k R \twoheadrightarrow L$ to their tensor product $(M \otimes_k N) \otimes_k R \twoheadrightarrow K \otimes_R L$. This is well-defined since the tensor product $\otimes$ preserves lines and epimorphisms. If $M = k^{(I)}$ and $N = k^{(J)}$, the Segre embedding is a map

$$\varsigma\colon \mathbb{P}^I_k \times \mathbb{P}^J_k \to \mathbb{P}^{I\times J}_k.$$

If $M = k^{m+1}$ and $N = k^{n+1}$, this becomes the map

$$\varsigma\colon \mathbb{P}^m_k \times \mathbb{P}^n_k \to \mathbb{P}^{mn+m+n}_k,$$

which is given in coordinates by the formula

$$\big([a_0 : \dots : a_m], [b_0 : \dots : b_n]\big) \mapsto [\text{all products } a_i b_j].$$

More generally, we have a Segre embedding

$$\varsigma\colon \prod_{i=1}^n \mathbb{P}(M_i) \to \mathbb{P}\Big(\bigotimes_{i=1}^n M_i\Big)$$

for any finite family of k-modules $M_1, \dots, M_n$. We claim that the Segre embedding is a closed immersion. Given surjective maps $M' \twoheadrightarrow M$ and $N' \twoheadrightarrow N$, we have a commutative square

$$\begin{array}{ccc} \mathbb{P}(M) \times \mathbb{P}(N) & \xrightarrow{\ \varsigma\ } & \mathbb{P}(M \otimes_k N) \\ \big\downarrow & & \big\downarrow \\ \mathbb{P}(M') \times \mathbb{P}(N') & \xrightarrow{\ \varsigma\ } & \mathbb{P}(M' \otimes_k N'), \end{array}$$

where the vertical maps are closed immersions. Hence, if the bottom horizontal map is a closed immersion, so is the top horizontal map. We may therefore assume that M and N have bases $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$, so that $M \otimes_k N$ has basis $(e_i \otimes f_j)_{i,j}$. For any map $\mathrm{Spec}(R) \to \mathbb{P}(M \otimes_k N)$, the pullback $(\mathbb{P}(M) \times \mathbb{P}(N)) \times_{\mathbb{P}(M\otimes_k N)} \mathrm{Spec}(R)$ satisfies Zariski descent, by Proposition 3.57 and Corollary 2.72. Using Proposition 2.79, we see that it is enough to show that each restriction

$$\varsigma_{ij} : \varsigma^{-1}(\mathrm{D}(e_i \otimes f_j)) \to \mathrm{D}(e_i \otimes f_j)$$

of ς is a closed immersion. The open subfunctor $\mathrm{D}(e_i \otimes f_j) \subset \mathbb{P}(M \otimes_k N)$ is an affine space on the set $(I \times J) - \{(i, j)\}$. Its preimage $\varsigma^{-1}(\mathrm{D}(e_i \otimes f_j)) \subset \mathbb{P}(M) \times \mathbb{P}(N)$ is the subfunctor $\mathrm{D}(e_i) \times \mathrm{D}(f_j)$, which is an affine space on the set $(I - \{i\}) \sqcup (J - \{j\})$. By inspection, the map ς_{ij} is then Spec of the map of polynomial rings

$$k\big[z_{kl} \bigm| (k,l) \in (I \times J) - \{(i, j)\}\big] \to k\big[x_k, y_l \bigm| k \in I - \{i\},\, l \in J - \{j\}\big], \quad z_{kl} \mapsto x_k y_l,$$

where $x_i = y_j = 1$. Since $z_{kj} \mapsto x_k$ and $z_{il} \mapsto y_l$, this map is surjective, so that $\varsigma_{i,j}$ is a closed immersion, as desired.

Corollary 3.96. *Let X and Y be projective k-schemes. Then $X \times Y$ is a projective k-scheme.*

Remark 3.97. The Veronese embedding of Example 3.68 is determined by the Segre embedding as follows: for any k-module M and $d \geq 1$, there is a commutative square

$$\begin{array}{ccc} \mathbb{P}(M) & \xrightarrow{\ \rho_d\ } & \mathbb{P}(\mathrm{Sym}^d_k(M)) \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow \\ \mathbb{P}(M)^{\times d} & \xrightarrow{\ \varsigma\ } & \mathbb{P}(M^{\otimes d}), \end{array}$$

where the vertical maps are the inclusions of the Σ_d-fixed points. This gives another proof that ρ_d is a closed immersion.

Next, we consider a generalization of projective spaces where we replace quotient lines with quotient spaces of arbitrary rank:

Definition 3.98 (Grassmannian)**.** Let k be a ring, M a k-module, and $n \in \mathbb{N}$. The rank n *Grassmannian* of M is the algebraic k-functor $\mathrm{Gr}_n(M)$ given by

$$\mathrm{Gr}_n(M)(R) = \{\text{quotient spaces of } M \otimes_k R \text{ of constant rank } n\}.$$

Remark 3.99. By definition, $\mathrm{Gr}_1(M) = \mathbb{P}(M)$. If M is a vector space of constant rank r, then duality induces isomorphisms $\mathrm{Gr}_n(M) \simeq \mathrm{Gr}_{r-n}(M^\vee)$ (Corollary 3.18).

Example 3.100 (The Plücker embedding)**.** Let M be a k-module and let $n \in \mathbb{N}$. The *Plücker embedding* is the map of algebraic k-functors

$$\varpi\colon \mathrm{Gr}_n(M) \to \mathbb{P}(\Lambda^n_k(M))$$

that sends a quotient space of rank n to its nth exterior power, which is a quotient line. We claim that the Plücker embedding is a closed immersion. For any surjective map $M' \twoheadrightarrow M$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_n(M) & \xrightarrow{\varpi} & \mathbb{P}(\Lambda^n_k(M)) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_n(M') & \xrightarrow{\varpi} & \mathbb{P}(\Lambda^n_k(M')), \end{array}$$

where the vertical maps are closed immersions. We can therefore assume that M has a basis $(e_i)_{i\in I}$, so that $\Lambda^n_k(M)$ has an induced basis $(e_J)_J$ (defined up to signs) indexed by the set $\binom{I}{n}$ of n-element subsets of I. Zariski descent for modules implies, as in Proposition 3.57, that $\mathrm{Gr}_n(M)$ satisfies Zariski descent. By Proposition 2.79, it thus suffices to show that each map $\varpi_J\colon \varpi^{-1}(\mathrm{D}(e_J)) \to \mathrm{D}(e_J)$ is a closed immersion. The open subfunctor $\mathrm{D}(e_J) \subset \mathbb{P}(\Lambda^n_k(M))$ is an affine space on the set $\binom{I}{n} - \{J\}$. The preimage $\varpi^{-1}(\mathrm{D}(e_J)) \subset \mathrm{Gr}_n(M)$ is the subfunctor consisting of quotient spaces $R^{(I)} \twoheadrightarrow V$ such that the composite $R^{(J)} \hookrightarrow R^{(I)} \twoheadrightarrow V$ is an isomorphism, which is an affine space on the set $(I - J) \times I$. By inspection, the map ϖ_J is $\mathbb{A}(-)$ of the map of free k-modules given by

$$e_K \mapsto \begin{cases} e_{(i,j)} & \text{if } K = (J - \{j\}) \cup \{i\}, \\ 0 & \text{else.} \end{cases}$$

As this map is evidently surjective, ϖ_J is a closed immersion, as desired. If M is of finite type, it follows that the Grassmannian $\mathrm{Gr}_n(M)$ is a projective k-scheme.

Definition 3.101 (Flag schemes)**.** Let M be a k-module and let $n = (n_1, \dots, n_s)$ be an increasing sequence of natural numbers.

(i) A *flag of type* n on M is a sequence of quotient modules

$$M \twoheadrightarrow F_s \twoheadrightarrow \cdots \twoheadrightarrow F_1$$

where each F_i is a vector space of constant rank n_i.

(ii) The type n *flag scheme* of M is the algebraic k-functor $\mathrm{Flag}_n(M)$ given by

$$\mathrm{Flag}_n(M)(R) = \{\text{flags of type } n \text{ on } M \otimes_k R\}.$$

Remark 3.102. By definition, $\mathrm{Flag}_{(n)}(M) = \mathrm{Gr}_n(M)$. If M is a vector space of constant rank r, then duality induces isomorphisms $\mathrm{Flag}_{(n_1,\dots,n_s)}(M) \simeq \mathrm{Flag}_{(r-n_s,\dots,r-n_1)}(M^\vee)$.

Example 3.103 (Projectivity of flag schemes)**.** Sending a flag to its components defines a monomorphism

$$\mathrm{Flag}_n(M) \hookrightarrow \prod_{i=1}^s \mathrm{Gr}_{n_i}(M).$$

We claim that it is a closed immersion, and hence that $\mathrm{Flag}_n(M)$ is a projective k-scheme if M is of finite type. Indeed, for any R-point $x\colon \mathrm{Spec}(R) \to \prod_{i=1}^s \mathrm{Gr}_{n_i}(M)$ classifying quotient spaces $F_1, \dots, F_s$ of $M \otimes_k R$, the preimage of $\mathrm{Flag}_n(M)$ by x is exactly the joint vanishing locus of the R-linear maps

$$\ker(M \otimes_k R \twoheadrightarrow F_{i+1}) \hookrightarrow M \otimes_k R \twoheadrightarrow F_i$$

for $1 \leq i \leq s-1$, which is a closed subfunctor of $\mathrm{Spec}(R)$ by Proposition 3.47(i).

3.11. **The projective Nullstellensatz.** Hilbert's Nullstellensatz (Theorem 2.89) implies an analogous result for solutions of homogeneous polynomial equations in projective space. We record this classical result here, although it is mostly of historical interest.

For a ring k, define the maps

$$\{\text{homogeneous subsets of } k[x_0, \dots, x_n]\} \underset{\mathrm{I}}{\overset{\mathrm{V}}{\rightleftarrows}} \{\text{subsets of } \mathbb{P}^n(k)\},$$

as follows:

$$\mathrm{V}(F) = \{x \in \mathbb{P}^n(k) \mid f(x) = 0 \text{ for all } f \in F\},$$
$$\mathrm{I}(X) = \{f \in k[x_0, \dots, x_n] \text{ homogeneous} \mid f(x) = 0 \text{ for all } x \in X\}.$$

Note that both maps are order-reversing and that $F \subset \mathrm{I}(\mathrm{V}(F))$ and $X \subset \mathrm{V}(\mathrm{I}(X))$ (in other words, this is an adjunction between posets). Note also that $\mathrm{I}(X)$ is always a saturated homogeneous ideal in $k[x_0, \dots, x_n]$, which is radical if k is reduced. Call a subset $X \subset \mathbb{P}^n(k)$ *algebraic* if it lies in the image of V, or equivalently if $X = \mathrm{V}(\mathrm{I}(X))$.

Proposition 3.104 (Projective Nullstellensatz)**.** *Let k be an algebraically closed field and let $n \in \mathbb{N}$. For any homogeneous subset $F \subset k[x_0, \dots, x_n]$, we have*

$$\mathrm{I}(\mathrm{V}(F)) = \sqrt{(F)}^{\,\mathrm{sat}}.$$

Consequently, the maps V *and* I *define a one-to-one correspondence*

$$\{\textit{saturated radical homogeneous ideals in } k[x_0, \dots, x_n]\} \underset{\mathrm{I}}{\overset{\mathrm{V}}{\rightleftarrows}} \{\textit{algebraic subsets of } \mathbb{P}^n(k)\}.$$

Corollary 3.105. *Let k be a field and let $n \in \mathbb{N}$. Then there is an order-reversing bijection*

$$\mathrm{V}\colon \{\textit{saturated radical homogeneous ideals in } k[x_0, \dots, x_n]\} \xrightarrow{\sim} \{\textit{vanishing loci in } \mathbb{P}^n\colon \mathrm{Field}_k^{\mathrm{fin}} \to \mathrm{Set}\}.$$

4. Quasi-coherent modules

4.1. Limits of categories.

Definition 4.1 (Category-valued functor)**.** Let $\mathcal{I}$ be a category. A *functor*

$$F\colon \mathcal{I} \to \mathrm{Cat}$$

consists of:

(i) for every object $I \in \mathcal{I}$, a category $F(I)$;
(ii) for every map $f\colon I \to J$ in $\mathcal{I}$, a functor $F(f)\colon F(I) \to F(J)$;
(iii) For every object $I \in \mathcal{I}$, a natural isomorphism $\eta_I\colon \mathrm{id}_{F(I)} \simeq F(\mathrm{id}_I)$;
(iv) for every pair of maps $f\colon I \to J$ and $g\colon J \to K$ in $\mathcal{I}$, a natural isomorphism $\mu_{f,g}\colon F(g) \circ F(f) \simeq F(g \circ f)$;

satisfying the following conditions:

(v) for every map $f\colon I \to J$ in $\mathcal{I}$, the following triangles commute:

$$\begin{array}{ccc} F(f) & \xrightarrow{\eta_I} & F(f) \circ F(\mathrm{id}_I) \\ & \searrow^{\mathrm{id}} & \downarrow \mu_{\mathrm{id}_I, f} \\ & & F(f), \end{array} \qquad \begin{array}{ccc} F(f) & \xrightarrow{\eta_J} & F(\mathrm{id}_J) \circ F(f) \\ & \searrow^{\mathrm{id}} & \downarrow \mu_{f, \mathrm{id}_J} \\ & & F(f); \end{array}$$

(vi) for every triple of maps $f\colon I \to J$, $g\colon J \to K$, and $h\colon K \to L$ in $\mathcal{I}$, the following square commutes:

$$\begin{array}{ccc} F(h) \circ F(g) \circ F(f) & \xrightarrow{\mu_{f,g}} & F(h) \circ F(g \circ f) \\ \mu_{g,h} \downarrow & & \downarrow \mu_{g \circ f, h} \\ F(h \circ g) \circ F(f) & \xrightarrow[\mu_{f, h \circ g}]{} & F(h \circ g \circ f). \end{array}$$

Example 4.2 (Self-indexing functor)**.** Let $\mathcal{C}$ be a category with pullbacks. There is then a functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}, \quad X \mapsto \mathcal{C}_{/X},$$

called the *self-indexing functor* of $\mathcal{C}$, sending a map $f\colon Y \to X$ in $\mathcal{C}$ to the pullback functor

$$f^*\colon \mathcal{C}_{/X} \to \mathcal{C}_{/Y}, \quad U \mapsto U \times_X Y.$$

The natural isomorphisms $\eta_X\colon \mathrm{id}_{\mathcal{C}_{/X}} \simeq \mathrm{id}_X^*$ and $\mu_{f,g}\colon f^* \circ g^* \simeq (g \circ f)^*$ are given by the canonical isomorphisms $U \simeq U \times_X X$ and $(U \times_X Y) \times_Y Z \simeq U \times_X Z$, induced by the universal property of pullbacks.

Example 4.3 (Categories of modules)**.** The assignment $R \mapsto \mathrm{Mod}_R$ is functorial in three ways:

(i) There is a functor
$$\mathrm{Mod}^*\colon \mathrm{CAlg} \to \mathrm{Cat}, \quad R \mapsto \mathrm{Mod}_R,$$
sending a ring map $f\colon R \to S$ to the extension of scalars functor
$$f^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto M \otimes_R S.$$
The natural isomorphisms $\eta_R\colon \mathrm{id}_{\mathrm{Mod}_R} \simeq \mathrm{id}_R^*$ and $\mu_{f,g}\colon g^* \circ f^* \simeq (g \circ f)^*$ are given by the canonical isomorphisms $M \simeq M \otimes_R R$ and $(M \otimes_R S) \otimes_S T \simeq M \otimes_R T$, induced by the universal property of scalar extension.

(ii) There is a functor
$$\mathrm{Mod}_*\colon \mathrm{CAlg}^{\mathrm{op}} \to \mathrm{Cat}, \quad R \mapsto \mathrm{Mod}_R,$$
sending a ring map $f\colon R \to S$ to the restriction of scalars functor
$$f_*\colon \mathrm{Mod}_S \to \mathrm{Mod}_R, \quad M \mapsto M.$$
In this case, the natural isomorphisms η_R and $\mu_{f,g}$ are taken to be the identity.

(iii) Finally, there is a functor
$$\mathrm{Mod}^!\colon \mathrm{CAlg} \to \mathrm{Cat}, \quad R \mapsto \mathrm{Mod}_R,$$
sending a ring map $f\colon R \to S$ to the coextension of scalars functor
$$f^!\colon \mathrm{Mod}_R \to \mathrm{Mod}_S, \quad M \mapsto \mathrm{Hom}_R(S, M).$$
As in (i), the natural isomorphisms η_R and $\mu_{f,g}$ are induced by the universal property of scalar coextension.

Example 4.4 (Categories of algebras)**.** The assignment $R \mapsto \mathrm{CAlg}_R$ is both covariant and contravariant in R as in Example 4.3(i,ii). There is however no analogue of Example 4.3(iii) for algebras.

Example 4.5 (Subcategories of modules)**.** Any property of modules that is preserved by extension of scalars defines a subfunctor of $\mathrm{Mod}^*\colon \mathrm{CAlg} \to \mathrm{Cat}$. This holds for the following properties, and any combination thereof: finitely generated, finitely presented, flat, projective, free, vector space, line, etc.

Example 4.6 (Posets of ideals)**.** If we view posets as categories, then a poset-valued functor $\mathcal{C} \to \mathrm{Pos}$ defines a category-valued functor (where η_I and $\mu_{f,g}$ are identities). For example, we have the following functors $\mathrm{CAlg} \to \mathrm{Pos}$:

(i) The poset Id_R of ideals in R is functorial in R: a map $R \to S$ induces $\mathrm{Id}_R \to \mathrm{Id}_S$, $I \mapsto IS$.

(ii) The poset Rad_R of radical ideals in R is functorial in R: a map $R \to S$ induces $\mathrm{Rad}_R \to \mathrm{Rad}_S$, $I \mapsto \sqrt{IS}$.

Notation 4.7. Let $\mathcal{I}$ be a category. The *left cone* $\mathcal{I}^{\triangleleft}$ on $\mathcal{I}$ is the category obtained from $\mathcal{I}$ by adjoining a new object $-\infty$ which is strictly initial, i.e., such that $\mathrm{Map}(-\infty, I) = *$ for all $I \in \mathcal{I}^{\triangleleft}$ and $\mathrm{Map}(I, -\infty) = \varnothing$ for all $I \in \mathcal{I}$. Dually, the *right cone* $\mathcal{I}^{\triangleright}$ is obtained from $\mathcal{I}$ by adding a strictly final object ∞.

Definition 4.8 (Limits of categories)**.** Let $F\colon \mathcal{I} \to \mathrm{Cat}$ be a functor. The *limit* of F, denoted by $\lim F$ or $\lim_{I \in \mathcal{I}} F(I)$, is the following category:

- An object of $\lim F$ consists of objects $x_I \in F(I)$ for all $I \in \mathcal{I}$ and isomorphisms $\alpha_f\colon F(f)(x_I) \xrightarrow{\sim} x_J$ for all $f\colon I \to J$ in $\mathcal{I}$ satisfying the *cocycle conditions*:
  (i) for every $I \in \mathcal{I}$, the following triangle commutes:
$$\begin{array}{ccc} x_I & \xrightarrow{\eta_I(x_I)} & F(\mathrm{id}_I)(x_I) \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \alpha_{\mathrm{id}_I}} \\ & & x_I. \end{array}$$
  (ii) for every pair of morphisms $f\colon I \to J$ and $g\colon J \to K$ in $\mathcal{I}$, the following square commutes:
$$\begin{array}{ccc} F(g)(F(f)(x_I)) & \xrightarrow{F(g)(\alpha_f)} & F(g)(x_J) \\ {\scriptstyle \mu_{f,g}(x_I)}\downarrow & & \downarrow{\scriptstyle \alpha_g} \\ F(g \circ f)(x_I) & \xrightarrow[\alpha_{g \circ f}]{} & x_K. \end{array}$$

- A morphism in $\lim F$ from (x, α) to (y, β) consists of morphisms $\varphi_I \colon x_I \to y_I$ for all $I \in \mathcal{I}$, such that for each map $f \colon I \to J$ in $\mathcal{I}$, the following square commutes:

$$\begin{array}{ccc} F(f)(x_I) & \xrightarrow{F(f)(\varphi_I)} & F(f)(y_J) \\ {\scriptstyle \alpha_f}\big\downarrow & & \big\downarrow{\scriptstyle \beta_f} \\ x_J & \xrightarrow[\varphi_J]{} & y_J. \end{array}$$

- Identities and compositions are defined pointwise.

By construction, there is an extension $\bar{F} \colon \mathcal{I}^{\triangleleft} \to \mathrm{Cat}$ of F with $\bar{F}(-\infty) = \lim F$. If $\bar{F} \colon \mathcal{I}^{\triangleleft} \to \mathrm{Cat}$ is any extension of F, then $\bar{F}$ induces a functor $\bar{F}(-\infty) \to \lim F$, and we say that $\bar{F}$ is a *limit diagram* if that functor is an equivalence.

Remark 4.9 (Colimits of categories)**.** Once limits have been defined, we can also define colimits: given $F \colon \mathcal{I} \to \mathrm{Cat}$, an extension $\bar{F} \colon \mathcal{I}^{\triangleright} \to \mathrm{Cat}$ is called a *colimit diagram* if, for every category $\mathcal{E}$, the functor

$$\mathrm{Fun}(\bar{F}(-), \mathcal{E}) \colon (\mathcal{I}^{\mathrm{op}})^{\triangleleft} \to \mathrm{Cat}$$

is a limit diagram. One can show that colimits of categories always exist, although they are difficult to describe explicitly in general.

Example 4.10 (Pullbacks of categories)**.** Given functors $\mathcal{C} \xrightarrow{f} \mathcal{E} \xleftarrow{g} \mathcal{D}$, the pullback $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ is the category whose objects are triples (x, y, α) with $x \in \mathcal{C}$, $y \in \mathcal{D}$, and $\alpha \colon f(x) \xrightarrow{\sim} g(y)$ in $\mathcal{E}$.

Example 4.11. Let $\mathcal{C}$ be a category and consider the diagram $\mathcal{C} \rightrightarrows \mathcal{C}$ where both functors are the identity. The limit of this diagram is equivalent to the category $\mathrm{Fun}(\mathrm{B}\mathbb{Z}, \mathcal{C})$ of $\mathbb{Z}$-equivariant objects of $\mathcal{C}$.

Example 4.12 (Zariski descent for modules)**.** Limits of categories allow us to reformulate Theorem 2.70 more succinctly: it is exactly the statement that, if $(f_i)_{i \in I}$ generates the unit ideal in R, then the diagram of categories

$$\mathrm{Mod}_R \to \prod_{i \in I} \mathrm{Mod}_{R_{f_i}} \rightrightarrows \prod_{i,j \in I} \mathrm{Mod}_{R_{f_i f_j}} \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i,j,k \in I} \mathrm{Mod}_{R_{f_i f_j f_k}}$$

obtained by restricting the functor Mod^* from Example 4.3(i) is a limit diagram. In the case $I = \{1, 2\}$, this is equivalent to the simpler statement that the square of categories

$$\begin{array}{ccc} \mathrm{Mod}_R & \longrightarrow & \mathrm{Mod}_{R_{f_1}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Mod}_{R_{f_2}} & \longrightarrow & \mathrm{Mod}_{R_{f_1 f_2}} \end{array}$$

is cartesian. Moreover, these results hold for any subfunctor of Mod^* defined by a Zariski-local property (as in Example 4.5), such as $R \mapsto \mathrm{Vect}_R$ and $R \mapsto \mathrm{Line}_R$.

4.2. **Quasi-coherence.**

Definition 4.13 (Quasi-coherent objects)**.** Let $F \colon \mathrm{CAlg} \to \mathrm{Cat}$ be a functor. For any algebraic functor X, we define the category $F(X)$ of *quasi-coherent F-objects* over X, or simply *F-objects* over X, as the limit

$$F(X) = \lim_{\mathrm{Spec}(R) \to X} F(R),$$

where the indexing category is (the opposite of) the category of elements of X. For example:

(i) The category of *quasi-coherent modules* over X is $\mathrm{Mod}_X = \lim_{\mathrm{Spec}(R) \to X} \mathrm{Mod}_R$.
(ii) The category of *quasi-coherent vector spaces* over X is $\mathrm{Vect}_X = \lim_{\mathrm{Spec}(R) \to X} \mathrm{Vect}_R$. These are called *vector bundles* over X.
(iii) The category of *quasi-coherent lines* over X is $\mathrm{Line}_X = \lim_{\mathrm{Spec}(R) \to X} \mathrm{Line}_R$. These are called *line bundles* over X.
(iv) The category of *quasi-coherent algebras* over X is $\mathrm{CAlg}_X = \lim_{\mathrm{Spec}(R) \to X} \mathrm{CAlg}_R$.
(v) The poset of *quasi-coherent ideals* over X is $\mathrm{Id}_X = \lim_{\mathrm{Spec}(R) \to X} \mathrm{Id}_R$.
(vi) The poset of *quasi-coherent radical ideals* over X is $\mathrm{Rad}_X = \lim_{\mathrm{Spec}(R) \to X} \mathrm{Rad}_R$.

Remark 4.14 (Functoriality of quasi-coherent objects)**.**

(i) The assignment $X \mapsto F(X)$ has a structure of functor $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{Cat}$. In particular, for every map $f \colon Y \to X$, we have a *base change* or *pullback* functor

$$f^* \colon F(X) \to F(Y),$$

given by precomposition with the induced functor $\mathrm{El}(Y)^{\mathrm{op}} \to \mathrm{El}(X)^{\mathrm{op}}$. The right adjoint to f^*, if it exists, is called the *pushforward* functor and denoted by f_*.

(ii) For $X = \mathrm{Spec}(R)$, evaluation at id_R induces an equivalence of categories $F(X) \xrightarrow{\sim} F(R)$. In other words, the construction of Definition 4.13 is an extension of F along the Yoneda embedding $\mathrm{CAlg} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}}$ (in fact, it is the *right Kan extension*). For example, a quasi-coherent module (algebra, ideal, etc.) over $\mathrm{Spec}(R)$ is the same as an R-module (algebra, ideal, etc.).

(iii) There is an obvious notion of a *natural transformation* between category-valued functors. Any natural transformation $\alpha\colon F \to G$ between functors $\mathrm{CAlg} \to \mathrm{Cat}$ extends automatically to a natural transformation between their quasi-coherent extensions $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{Cat}$. In particular, for any algebraic functor X, there is an induced functor $\alpha_X\colon F(X) \to G(X)$.

Example 4.15 (Quasi-coherent modules)**.** Let us spell out explicitly the definition of a quasi-coherent module. A quasi-coherent module M over an algebraic functor X consists of the following data:

(i) for every $x\colon \mathrm{Spec}(R) \to X$, an R-module $M(x)$;

(ii) for every commutative triangle

$$\begin{array}{ccc} \mathrm{Spec}(S) & & \\ {\scriptstyle \mathrm{Spec}(\varphi)}\big\downarrow & \overset{y}{\searrow} & X, \\ \mathrm{Spec}(R) & \underset{x}{\nearrow} & \end{array}$$

an S-linear isomorphism $\varphi^*(M(x)) \simeq M(y)$;

such that the isomorphisms in (ii) satisfy the cocycle conditions of Definition 4.8. Under the equivalence $\mathrm{Mod}_{\mathrm{Spec}(R)} \simeq \mathrm{Mod}_R$ of Remark 4.14(ii), the R-module $M(x)$ in (i) is identified with the pullback $x^*(M)$, and the isomorphism in (ii) is an instance of the natural isomorphism $\mu_{\mathrm{Spec}(\varphi),x}\colon \mathrm{Spec}(\varphi)^* \circ x^* \simeq y^*$, which is part of the functor $\mathrm{Mod}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{Cat}$ of Remark 4.14(i).

Remark 4.16 (Pushforward of quasi-coherent modules)**.** Let $f\colon Y \to X$ be a map of algebraic functors. For quasi-coherent modules and algebras, the pushforward functor f_* always exists, modulo "size issues" as in Remark 2.53. That is, if M is a quasi-coherent module over Y, then $f_*(M)$ is potentially a large quasi-coherent module over X. This does not happen if X and Y are accessible, hence does not happen in practice.

Example 4.17 (Set-valued functors)**.** As a special case, we can apply Definition 4.13 to any set-valued functor $F\colon \mathrm{CAlg} \to \mathrm{Set}$. By Corollary 2.11, we then have $F(X) = \mathrm{Map}(X, F)$ for any algebraic functor X. In other words, the quasi-coherent extension of F is $\mathrm{Map}(-, F)$.

Remark 4.18. Let $F\colon \mathrm{CAlg} \to \mathrm{Cat}$ be a functor. Any properties or constructions within the categories $F(R)$ that are compatible with base change automatically extend to the categories of quasi-coherent F-objects. For example:

(i) (Properties of modules) A quasi-coherent module can be *of finite type*, *of finite presentation*, *flat*, *projective*, *free*, etc. Vector bundles and line bundles form full subcategories the category of quasi-coherent modules: $\mathrm{Line}_X \subset \mathrm{Vect}_X \subset \mathrm{Mod}_X$.

(ii) (Properties of module maps) A morphism of quasi-coherent modules can be *zero*, *surjective*, *universally injective*, etc. On the other hand, injectivity does not a priori make sense, as this property is not preserved by base change.

(iii) (Sums of module maps) Two morphisms $f, g\colon M \to N$ in Mod_X have a *sum* $f + g\colon M \to N$. This defines an abelian group structure on $\mathrm{Map}_{\mathrm{Mod}_X}(M, N)$.

(iv) (Colimits of modules) The category of quasi-coherent modules over X has colimits, which are computed pointwise: $(\mathrm{colim}_i\, M_i)(x) = \mathrm{colim}_i\, M_i(x)$. On the other hand, while limits exist if X is accessible, they need not be computed pointwise.

(v) (Tensor products of modules) The category of quasi-coherent modules over X has a symmetric monoidal structure with $(M \otimes N)(x) = M(x) \otimes N(x)$.

(vi) (Powers of modules) For any $d \in \mathbb{N}$, there are symmetric, exterior, and divided power functors $\mathrm{Sym}^d, \Lambda^d, \Gamma^d\colon \mathrm{Mod}_X \to \mathrm{Mod}_X$.

(vii) (Operations with ideals) Quasi-coherent ideals $I, J \in \mathrm{Id}_X$ have a *sum* $I + J$ and a *product* IJ, defined by $(I + J)(x) = I(x) + J(x)$ and $(IJ)(x) = I(x)J(x)$.

(viii) (Radicals of ideals) Any quasi-coherent ideal $I \in \mathrm{Id}_X$ has a *radical* $\sqrt{I} \in \mathrm{Rad}_X$ defined by $(\sqrt{I})(x) = \sqrt{I(x)}$.

(ix) (Properties of algebras) A quasi-coherent algebra can be *of finite type*, *of finite presentation*, *finite*, *flat*, *free*, etc. A quasi-coherent $\mathbb{N}$-graded algebra A can be *generated by* A_1, *finitely generated as an* A_0*-algebra*, etc.

(x) (Properties of algebra maps) A morphism of quasi-coherent algebras can be *surjective*, a *localization*, have a *nilpotent kernel*, etc. A morphism of quasi-coherent $\mathbb{N}$-graded algebras can also be *eventually surjective*.

(xi) (Underlying modules and symmetric algebras) There is a forgetful functor $\mathrm{CAlg}_X \to \mathrm{Mod}_X$ with a left adjoint Sym, such that the underlying quasi-coherent module of $\mathrm{Sym}(M)$ is $\bigoplus_{d\in\mathbb{N}} \mathrm{Sym}^d(M)$.

(xii) etc.

Warning 4.19. Beware that the forgetful functors $\mathrm{Rad}_R \hookrightarrow \mathrm{Id}_R \to \mathrm{Mod}_R$ are *not* compatible with base change, and so do not extend to arbitrary algebraic functors X (although they will when X is a scheme): a quasi-coherent radical ideal over X need not have an underlying quasi-coherent ideal, and a quasi-coherent ideal over X need not have an underlying quasi-coherent module.

Notation 4.20 (The quasi-coherent algebra of functions)**.** Let X be an algebraic functor. We denote by $\mathcal{O}_X \in \mathrm{CAlg}_X$ the quasi-coherent algebra given by $\mathcal{O}_X(x\colon \mathrm{Spec}(R) \to X) = R$. It is the initial object in CAlg_X, and its underlying quasi-coherent module is the unit of the tensor product in Mod_X. For any map $f\colon Y \to X$, there is a canonical isomorphism $f^*(\mathcal{O}_X) \simeq \mathcal{O}_Y$ in CAlg_Y.

Definition 4.21 (Global sections of a quasi-coherent module)**.** Let X be an algebraic functor and M a quasi-coherent module over X. A *global section* of M is a map $\mathcal{O}_X \to M$ in Mod_X. We denote by $M(X)$ or $\Gamma(X, M)$ the abelian group of global sections of M. By definition of Mod_X, it is the limit

$$M(X) = \Gamma(X, M) = \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} \mathrm{Map}_{\mathrm{Mod}_R}(R, M(x)) = \lim_{(R,x)\in \mathrm{El}(X)^{\mathrm{op}}} M(x).$$

Remark 4.22 (Functions as global sections)**.** A global section of $\mathcal{O}_X$ is exactly a function on X in the sense of Definition 2.51: $\Gamma(X, \mathcal{O}_X) = \mathcal{O}(X)$. Hence, for any $M \in \mathrm{Mod}_X$, the abelian group $M(X)$ has a canonical structure of module over the ring of functions $\mathcal{O}(X)$. We can also see this module structure as follows: if $a\colon X \to \mathrm{Spec}(\mathcal{O}(X))$ is the canonical map (see Remark 2.54), then $M(X) = a_*(M)$ as an $\mathcal{O}(X)$-module.

4.3. Classification of closed and open subfunctors.

Proposition 4.23 (Classification of closed and open subfunctors)**.** *Let X be an algebraic functor.*

(i) *There is an isomorphism of posets*

$$\mathrm{V}\colon \{\textit{quasi-coherent ideals over } X\}^{\mathrm{op}} \xrightarrow{\sim} \{\textit{closed subfunctors of } X\}.$$

(ii) *There is an isomorphism of posets*

$$\mathrm{D}\colon \{\textit{quasi-coherent radical ideals over } X\} \xrightarrow{\sim} \{\textit{open subfunctors of } X\}.$$

Definition 4.24 (Open complement)**.** Let $Z \subset X$ be a closed subfunctor defined by the quasi-coherent ideal I. The *open complement* $X - Z$ of Z in X is the open subfunctor defined by the quasi-coherent radical ideal $\sqrt{I}$.

Example 4.25.

(i) If A is a ring and $F \subset A$ is a subset, then $\mathrm{D}(F)$ is the open complement of $\mathrm{V}(F)$ in $\mathrm{Spec}(A)$. For example, the punctured affine space $\mathbb{A}^I - 0$ (Example 2.57) is the open complement of 0 in $\mathbb{A}^I$.

(ii) If A is an $\mathbb{N}$-graded ring and $F \subset A$ is a homogeneous subset, then $\mathrm{D}(F)$ is the open complement of $\mathrm{V}(F)$ in $\mathrm{Proj}(A)$. For example, in $\mathbb{P}^{I\sqcup\{0\}}$, the affine space $\mathrm{D}(x_0) \simeq \mathbb{A}^I$ is the open complement of the hyperplane at infinity $\mathrm{V}(x_0) \simeq \mathbb{P}^I$.

Warning 4.26. Different closed subfunctors can have the same open complement, since different ideals can have the same radical (e.g., (x) and (x^2) in $\mathbb{Z}[x]$). For this reason, there is no notion of "closed complement" of an open subfunctor.

Example 4.27 (Closed image)**.** Any map of rings $\varphi\colon A \to B$ admits a canonical factorization $A \twoheadrightarrow A/\ker(\varphi) \hookrightarrow B$, so that $\mathrm{V}(\ker(\varphi)) \subset \mathrm{Spec}(A)$ is the smallest closed subfunctor through which $\mathrm{Spec}(\varphi)$ factors. This factorization generalizes to algebraic functors as follows. Let $f\colon Y \to X$ be a map of algebraic functors. The *closed image* of f, also called the *scheme-theoretic image*, is the smallest closed subfunctor of X through which f factors. By Proposition 4.23(i), the closed image of f is $\mathrm{V}(I) \subset X$, where I is the largest quasi-coherent ideal such that $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ factors through $\mathcal{O}_X/I$ in CAlg_X.

Remark 4.28 (Loci associated with maps of quasi-coherent modules)**.** The loci from Definition 3.44 have a direct generalization to any algebraic functor X: a map $f\colon M \to N$ in Mod_X defines subfunctors $\mathrm{V}(f)$, $\mathrm{Epi}(f)$, $\mathrm{Mono}(f)$, and $\mathrm{Iso}(f)$ of X. These are characterized by the property that their preimage along any R-point $x\colon \mathrm{Spec}(R) \to X$ is the corresponding locus for the R-linear map $f(x)\colon M(x) \to N(x)$. Proposition 3.47 then immediately generalizes to this setting: if N is a vector bundle, then $\mathrm{V}(f)$ is a closed subfunctor and $\mathrm{Epi}(f)$ an open subfunctor of X, and if both M and N are vector bundles, then also $\mathrm{Mono}(f)$ and $\mathrm{Iso}(f)$ are open subfunctors of X.

Call a subfunctor *clopen* if it is both closed and open. Recall that $\mathrm{Idem}(R)$ is the set of idempotent elements in a ring R.

Proposition 4.29 (Classification of clopen subfunctors)**.** *For any algebraic functor X, there is a bijection*

$$\mathrm{Idem}(\mathcal{O}(X)) \xrightarrow{\sim} \{\textit{clopen subfunctors of } X\}, \quad e \mapsto \mathrm{V}(e) = \mathrm{D}(1-e).$$

Remark 4.30. By Example 2.31 and Remark 2.54, there is a bijection

$$\mathrm{Map}(X, \mathrm{Spec}(\mathbb{Z}\times\mathbb{Z})) \xrightarrow{\sim} \mathrm{Idem}(\mathcal{O}(X)), \quad f \mapsto f^*(1,0).$$

Thus, clopen subfunctors are also in bijection with maps to $\mathrm{Spec}(\mathbb{Z}\times\mathbb{Z})$, which is the coproduct $*\sqcup *$ in the category of affine schemes. This is analogous to the following (much more obvious) topological statement: if T is a topological space, there is a bijection between clopen subsets of T and continuous maps $T \to * \sqcup *$.

4.4. **Relative** Spec **and** Proj**.** The constructions Spec and Proj are compatible with base change in the following sense. If A is an R-algebra (resp. an $\mathbb{N}$-graded R-algebra) and $R \to S$ is a ring map, then the following squares are cartesian:

$$\begin{array}{ccc} \mathrm{Spec}(A\otimes_R S) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R), \end{array} \qquad \begin{array}{ccc} \mathrm{Proj}(A\otimes_R S) & \longrightarrow & \mathrm{Proj}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(S) & \longrightarrow & \mathrm{Spec}(R). \end{array}$$

This allows us to extend Spec and Proj to quasi-coherent algebras:

Definition 4.31 (Relative Spec and Proj)**.** Let X be an algebraic functor.

(i) Let A be a quasi-coherent algebra over X. The algebraic functor $\mathrm{Spec}(A)$ over X is defined by

$$\mathrm{Spec}(A)(R) = \{(x,a) \mid x \in X(R) \text{ and } a \text{ is a section of } \mathrm{Spec}(A(x)) \to \mathrm{Spec}(R)\}.$$

This defines a functor $\mathrm{Spec}\colon \mathrm{CAlg}_X^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg},\mathrm{Set})_{/X}$.

(ii) Let A be a quasi-coherent $\mathbb{N}$-graded algebra over X. The algebraic functor $\mathrm{Proj}(A)$ over X is defined by

$$\mathrm{Proj}(A)(R) = \{(x,a) \mid x \in X(R) \text{ and } a \text{ is a section of } \mathrm{Proj}(A(x)) \to \mathrm{Spec}(R)\}.$$

This defines a functor $\mathrm{Proj}\colon (\mathrm{CAlg}_X^{\mathbb{N},\mathrm{es}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg},\mathrm{Set})_{/X}$.

Definition 4.32 (Relative affine and projective spaces)**.** Let X be an algebraic functor and let M be a quasi-coherent module over X.

(i) The *affine space* $\mathbb{A}(M)$ over X is $\mathrm{Spec}(\mathrm{Sym}(M))$. Explicitly:

$$\mathbb{A}(M)(R) = \{(x,a) \mid x\in X(R) \text{ and } a\colon M(x)\to R \text{ is an } R\text{-linear map}\}.$$

(ii) The *punctured affine space* $\mathbb{A}(M)-0$ over X is $\mathrm{D}(M) \subset \mathrm{Spec}(\mathrm{Sym}(M))$. Explicitly:

$$(\mathbb{A}(M)-0)(R) = \{(x,a) \mid x\in X(R) \text{ and } a\colon M(x)\twoheadrightarrow R \text{ is a surjective } R\text{-linear map}\}.$$

(iii) The *projective space* $\mathbb{P}(M)$ over X is $\mathrm{Proj}(\mathrm{Sym}(M))$. Explicitly:

$$\mathbb{P}(M)(R) = \{(x,a) \mid x\in X(R) \text{ and } a\colon M(x)\twoheadrightarrow L \text{ is a quotient } R\text{-line}\}.$$

Remark 4.33 (Morphisms into Spec and Proj)**.** Using Corollary 2.11, we obtain the following descriptions of maps into $\mathrm{Spec}(A)$ or $\mathrm{Proj}(A)$. Let X and Y be algebraic functors.

(i) For any $A \in \mathrm{CAlg}_X$, a map $Y \to \mathrm{Spec}(A)$ consists of a map $f\colon Y\to X$ and a map $f^*(A) \to \mathcal{O}_Y$ in CAlg_Y (cf. Remark 2.54).

(ii) Let A be a quasi-coherent $\mathbb{N}$-graded algebra over X generated by A_1. Then a map $Y \to \mathrm{Proj}(A)$ consists of a map $f\colon Y\to X$ and a quotient line bundle $f^*(A_1) \twoheadrightarrow L$ in Mod_Y such that the induced map $\mathrm{Sym}(f^*(A_1)) \twoheadrightarrow \mathrm{Sym}(L)$ factors through $f^*(A)$.

The relative Spec and Proj functors allow us to relativize the notions of affine, quasi-affine, projective, and quasi-projective schemes:

Definition 4.34 (Affine and quasi-affine morphisms)**.** Let $f\colon Y \to X$ be a map of algebraic functors.

(i) We say that f is *affine*, or that Y is affine over X, if there exists $A \in \mathrm{CAlg}_X$ and an isomorphism $Y \simeq \mathrm{Spec}(A)$ over X. We denote by Aff_X the category of algebraic functors affine over X.
(ii) We say that f is *quasi-affine*, or that Y is quasi-affine over X, if there exists $V \in \mathrm{Aff}_X$, $I \in \mathrm{Rad}_V^{\mathrm{ft}}$, and an isomorphism $Y \simeq \mathrm{D}(I)$ over X. We denote by QAff_X the category of algebraic functors quasi-affine over X.

Definition 4.35 (Projective and quasi-projective morphisms)**.** Let $f\colon Y \to X$ be a map of algebraic functors.

(i) We say that f is *projective*, or that Y is projective over X, if there exists $A \in \mathrm{CAlg}_X^{\mathbb{N},\mathrm{es}}$, generated by A_1 and finitely generated, and an isomorphism $Y \simeq \mathrm{Proj}(A)$ over X. We denote by Proj_X the category of algebraic functors projective over X.
(ii) We say that f is *quasi-projective*, or that Y is quasi-projective over X, if there exists $V \in \mathrm{Proj}_X$, $I \in \mathrm{Rad}_V^{\mathrm{ft}}$, and an isomorphism $Y \simeq \mathrm{D}(I)$ over X. We denote by QProj_X the category of algebraic functors quasi-projective over X.

Affineness turns out to be a "quasi-coherent property" in the sense that the functor $X \mapsto \mathrm{Aff}_X$ is right Kan extended from affine schemes:

Proposition 4.36 (Characterization of affine morphisms)**.** *Let X be an algebraic functor.*

(i) *A map $Y \to X$ is affine if and only if, for every ring R and every map $\mathrm{Spec}(R) \to X$, the pullback $Y \times_X \mathrm{Spec}(R)$ is affine*
(ii) *The functor* Spec *defines an equivalence of categories*

$$\mathrm{Spec}\colon \mathrm{CAlg}_X^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Aff}_X,$$

whose inverse sends $f\colon Y \to X$ to $f_*(\mathcal{O}_Y)$.

Warning 4.37. The analogue of Proposition 4.36(i) does not hold for quasi-affine, projective, and quasi-projective morphisms.

Example 4.38 (Closed immersions as affine morphisms)**.** By Proposition 4.36, any closed immersion $i\colon Z \hookrightarrow X$ is affine, and $Z \simeq \mathrm{Spec}(i_*(\mathcal{O}_Z))$. Moreover, an affine map $f\colon Y \to X$ is a closed immersion if and only if the induced map $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is surjective.

4.5. **Modules over quasi-projective schemes.**

Definition 4.39 (Zariski-local epimorphism)**.** A map of algebraic functors $Y \to X$ is called a *Zariski-local epimorphism* if for every $\mathrm{Spec}(R) \to X$, there exists $f_1, \dots, f_n \in R$ generating the unit ideal such that each composite $\mathrm{Spec}(R_{f_i}) \hookrightarrow \mathrm{Spec}(R) \to X$ lifts to Y.

Example 4.40.

(i) Any epimorphism is a Zariski-local epimorphism. In particular, any map with a section, such as the projection $\mathbb{A}^n \times X \to X$, is a Zariski-local epimorphism.
(ii) Let $(F_i)_{i\in I}$ be a family of subsets of $\mathcal{O}(X)$ whose union generates the unit ideal. Then the map $\coprod_{i\in I} \mathrm{D}(F_i) \to X$ is a Zariski-local epimorphism.
(iii) Let L be a line bundle over X. A family $(s_i)_{i\in I}$ of global sections of L is called *generating* if, for every $x \in \mathrm{El}(X)$, the family $(s_i(x))_{i\in I}$ generates the line $L(x)$. In this case, the map $\coprod_{i\in I} \mathrm{D}(s_i) \to X$ is a Zariski-local epimorphism.
(iv) The map $\mathbb{A}^I - 0 \to \mathbb{P}^I$ is a Zariski-local epimorphism, by Proposition 3.24. More generally, for any $\mathbb{N}$-graded ring A, the map $\mathrm{D}(A_1) \to \mathrm{Proj}_1(A)$ is a Zariski-local epimorphism (Remark 3.63).

Proposition 4.41 (Zariski descent for quasi-coherent modules)**.** *Let $(Y_i \to X)_{i\in I}$ be a family of maps of algebraic functors such that $\coprod_{i\in I} Y_i \to X$ is a Zariski-local epimorphism. Then the following diagram of categories is a limit diagram:*

$$\mathrm{Mod}_X \to \prod_{i\in I} \mathrm{Mod}_{Y_i} \rightrightarrows \prod_{i,j\in I} \mathrm{Mod}_{Y_i\times_X Y_j} \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \prod_{i,j,k\in I} \mathrm{Mod}_{Y_i\times_X Y_j\times_X Y_k}.$$

In particular, the functor $\mathrm{Mod}_X \to \prod_{i\in I} \mathrm{Mod}_{Y_i}$ is conservative.

Remark 4.42. Proposition 4.41 is a formal consequence of the Zariski descent property of Mod^* from Example 4.12: for any functor $F\colon \mathrm{CAlg} \to \mathrm{Cat}$ that satisfies the latter property, its quasi-coherent extension $X \mapsto F(X)$ also satisfies Proposition 4.41. For example, there are analogous statements for quasi-coherent algebras, ideals, and radical ideals.

Corollary 4.43 (Zariski descent for functions)**.** *Let $(Y_i \to X)_{i\in I}$ be a family of maps of algebraic functors such that $\coprod_{i\in I} Y_i \to X$ is a Zariski-local epimorphism. Then the following diagram of rings is an equalizer:*

$$\mathcal{O}(X) \to \prod_{i\in I} \mathcal{O}(Y_i) \rightrightarrows \prod_{i,j\in I} \mathcal{O}(Y_i \times_X Y_j).$$

For the following examples, recall the notion of I-local module and the associated localization functor $\mathrm{L}_I\colon \mathrm{Mod}_A \to \mathrm{Mod}_A^{I\text{-loc}}$ (Definition 3.74).

Example 4.44 (Modules over nonvanishing loci)**.** Let A be a ring and $I \subset A$ a subset. Then the map $\coprod_{f\in I} \mathrm{Spec}(A_f) \to \mathrm{D}(I)$ is a Zariski-local epimorphism. By Proposition 4.41, there is a limit diagram

$$\mathrm{Mod}_{\mathrm{D}(I)} \to \prod_{f\in I} \mathrm{Mod}_{A_f} \rightrightarrows \prod_{f,g\in I} \mathrm{Mod}_{A_{fg}} \mathrel{\substack{\to\\\to\\\to}} \prod_{f,g,h\in I} \mathrm{Mod}_{A_{fgh}}.$$

In particular, the pullback functor $\mathrm{Mod}_A \to \mathrm{Mod}_{\mathrm{D}(I)}$ factors through the localization functor L_I.

Example 4.45 (Modules over Proj)**.** Let A be an $\mathbb{N}$-graded ring.

(i) Let $I \subset A_+$ be a homogeneous subset such that $A_+ \subset \sqrt{(I)}$. Then the map $\coprod_{f\in I} \mathrm{Spec}(A_{(f)}) \to \mathrm{Proj}(A)$ is a Zariski-local epimorphism. Proposition 4.41 gives the limit diagram

$$\mathrm{Mod}_{\mathrm{Proj}(A)} \to \prod_{f\in I} \mathrm{Mod}_{A_{(f)}} \rightrightarrows \prod_{f,g\in I} \mathrm{Mod}_{A_{(fg)}} \mathrel{\substack{\to\\\to\\\to}} \prod_{f,g,h\in I} \mathrm{Mod}_{A_{(fgh)}}.$$

(ii) Consider $\mathrm{D}(A_1) \subset \mathrm{Spec}(A)$. By Remark 3.63, the canonical map $\mathrm{D}(A_1) \to \mathrm{Proj}_1(A)$ is a Zariski-local epimorphism, and it induces a monomorphism $\mathrm{D}(A_1)/\mathbb{G}_{\mathrm{m}} \hookrightarrow \mathrm{Proj}_1(A)$. Hence, the n-fold fiber product of $\mathrm{D}(A_1)$ over $\mathrm{Proj}_1(A)$ is the same as its n-fold fiber product over $\mathrm{D}(A_1)/\mathbb{G}_{\mathrm{m}}$, which is $\mathrm{D}(A_1) \times \mathbb{G}_{\mathrm{m}}^{n-1}$ since the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathrm{D}(A_1)$ is free. By Proposition 4.41, we get a limit diagram of the form

$$\mathrm{Mod}_{\mathrm{Proj}_1(A)} \to \mathrm{Mod}_{\mathrm{D}(A_1)} \rightrightarrows \mathrm{Mod}_{\mathrm{D}(A_1)\times\mathbb{G}_{\mathrm{m}}} \mathrel{\substack{\to\\\to\\\to}} \mathrm{Mod}_{\mathrm{D}(A_1)\times\mathbb{G}_{\mathrm{m}}\times\mathbb{G}_{\mathrm{m}}}.$$

Example 4.46 (Modules over $\mathbb{P}^1$)**.** In the case of $\mathbb{P}^1_k = \mathrm{Proj}(k[x,y])$, Example 4.45(i) applies with the two-element set $I = \{x, y\}$. We have $k[x,y]_{(x)} = k[u]$ with $u = y/x$, $k[x,y]_{(y)} = k[v]$ with $v = x/y$, and $k[x,y]_{(xy)} = k[t^{\pm 1}]$ with $t = y/x$. Hence, the limit diagram may be rewritten as a cartesian square

$$\begin{array}{ccc} \mathrm{Mod}_{\mathbb{P}^1_k} & \longrightarrow & \mathrm{Mod}_{k[u]} \\ \downarrow & & \downarrow{\scriptstyle u\mapsto t} \\ \mathrm{Mod}_{k[v]} & \xrightarrow{v\mapsto t^{-1}} & \mathrm{Mod}_{k[t^{\pm 1}]}. \end{array}$$

A quasi-coherent module over $\mathbb{P}^1_k$ is therefore a triple (M, N, α), where M is a $k[u]$-module, N is a $k[v]$-module, and $\alpha\colon M[u^{-1}] \xrightarrow{\sim} N[v^{-1}]$ is an isomorphism of $k[t^{\pm 1}]$-modules (i.e., an isomorphism of k-modules such that the action of u on $M[u^{-1}]$ is *inverse* to the action of v on $N[v^{-1}]$).

Example 4.47 (Modules over the affine line with doubled origin)**.** Let X be the affine line with doubled origin over k, which is the algebraic k-functor defined by

$$X(R) = \{(f, e) \mid f \in R,\ e \in R/(f), \text{ and } e^2 = e\}.$$

Let $X_0 \subset X$ and $X_1 \subset X$ be the loci where $e = 0$ and $e = 1$, respectively. One can check that:

- X_0 and X_1 are open subfunctors of X such that $X_0 \sqcup X_1 \to X$ is a Zariski-local epimorphism;
- the map $X \to \mathbb{A}^1_k$, $(f, e) \mapsto f$, restricts to isomorphisms

$$X_0 \xrightarrow{\sim} \mathbb{A}^1_k, \quad X_1 \xrightarrow{\sim} \mathbb{A}^1_k, \quad \text{and} \quad X_0 \cap X_1 \xrightarrow{\sim} \mathbb{G}_{\mathrm{m},k}.$$

By Proposition 4.41, we obtain a cartesian square

$$\begin{array}{ccc} \mathrm{Mod}_X & \longrightarrow & \mathrm{Mod}_{k[x]} \\ \downarrow & & \downarrow{\scriptstyle x\mapsto x} \\ \mathrm{Mod}_{k[x]} & \xrightarrow{x\mapsto x} & \mathrm{Mod}_{k[x^{\pm 1}]}. \end{array}$$

A quasi-coherent module over X is therefore a triple (M, N, α), where M is a $k[x]$-module, N is a $k[x]$-module, and $\alpha\colon M[x^{-1}] \xrightarrow{\sim} N[x^{-1}]$ is an isomorphism of $k[x^{\pm 1}]$-modules (or equivalently of $k[x]$-modules).

Example 4.48 (Modules over Grassmannians)**.** Let M be a k-module and let $n \in \mathbb{N}$. Let $\mathrm{St}_n(M)$ be the algebraic k-functor given by

$$\mathrm{St}_n(M)(R) = \{\text{surjective } R\text{-linear maps } M \otimes_k R \to R^n\},$$

called the *Stiefel scheme* of M. By Proposition 3.47(ii), this is an open subfunctor of the affine space $\mathbb{A}(M^n)$. There is an obvious map $\mathrm{St}_n(M) \to \mathrm{Gr}_n(M)$, which is a Zariski-local epimorphism since vector spaces are Zariski-locally free. Moreover, there is a free action of the affine group scheme GL_n on St_n, such that $\mathrm{St}_n(M) \to \mathrm{Gr}_n(M)$ is GL_n-invariant and induces a monomorphism $\mathrm{St}_n(M)/\mathrm{GL}_n \hookrightarrow \mathrm{Gr}_n(M)$; in the case $n = 1$, this recovers the known monomorphism $(\mathbb{A}(M) - 0)/\mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{P}(M)$. As in Example 4.45(ii), we then obtain a limit diagram of the form

$$\mathrm{Mod}_{\mathrm{Gr}_n(M)} \to \mathrm{Mod}_{\mathrm{St}_n(M)} \rightrightarrows \mathrm{Mod}_{\mathrm{St}_n(M)\times \mathrm{GL}_n} \substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow} \mathrm{Mod}_{\mathrm{St}_n(M)\times \mathrm{GL}_n \times \mathrm{GL}_n},$$

where each term can be further expanded using Example 4.44.

Using Example 4.44, we can identify quasi-coherent modules over quasi-affine schemes:

Theorem 4.49 (Quasi-coherent modules over quasi-affine schemes)**.** *Let A be a ring and let $I \subset A$ be a finite subset with nonvanishing locus $\mathrm{D}(I) \subset \mathrm{Spec}(A)$. Then the pullback functor $\mathrm{Mod}_A \to \mathrm{Mod}_{\mathrm{D}(I)}$ induces an equivalence of categories*

$$\mathrm{Mod}_A^{I\text{-loc}} \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{D}(I)}.$$

Corollary 4.50 (Affine completion of quasi-affine schemes)**.** *Let X be a quasi-affine scheme. Then the canonical map $X \to \mathrm{Spec}(\mathcal{O}(X))$ is an open immersion defined by a finitely generated radical ideal.*

Construction 4.51 (Quasi-coherent module associated with a $\mathbb{Z}$-graded module)**.** Let A be an $\mathbb{N}$-graded ring. We define a functor

$$\{\mathbb{Z}\text{-graded } A\text{-modules}\} \to \mathrm{Mod}_{\mathrm{Proj}(A)}, \quad M \mapsto \tilde{M},$$

as follows. Let M be a $\mathbb{Z}$-graded A-module and let $x\colon \mathrm{Spec}(R) \to \mathrm{Proj}(A)$ classify a quotient $\mathbb{N}$-graded algebra $\varphi\colon A^{(d)} \otimes_{A_0} R \twoheadrightarrow \mathrm{Sym}_R(L)$ for some $d \geq 1$ (if A is generated by $A_{\leq 1}$, we can take $d = 1$). Let $\tilde{\varphi}$ be the induced map of $\mathbb{Z}$-graded rings $A^{(d)} \to \bigoplus_{n\in\mathbb{Z}} L^{\otimes n}$. We set

$$\tilde{M}(x) = \tilde{\varphi}^*(M^{(d)})_0 \in \mathrm{Mod}_R.$$

Note that for every $d \geq 1$ and $f \in A_d$, the pullback functor $\mathrm{Mod}_{\mathrm{Proj}(A)} \to \mathrm{Mod}_{A_{(f)}}$ sends $\tilde{M}$ to $M_{(f)}$. In particular, by Example 4.45(i), the functor $M \mapsto \tilde{M}$ factors through the localization functor L_{A_+}.

Using either (i) or (ii) of Example 4.45, we can make the following computation:

Theorem 4.52 (Quasi-coherent modules over quasi-projective schemes)**.** *Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$ and $I \subset A_1$ a finite subset with nonvanishing locus $\mathrm{D}(I) \subset \mathrm{Proj}(A)$. Then the functor $M \mapsto \tilde{M}$ induces an equivalence*

$$\{I\text{-local } \mathbb{Z}\text{-graded } A\text{-modules}\} \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{D}(I)}.$$

In particular, if A is a finitely generated A_0-algebra, then

$$\{A_1\text{-local } \mathbb{Z}\text{-graded } A\text{-modules}\} \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{Proj}(A)}.$$

4.6. **Serre twists.** Let M be a $\mathbb{Z}$-graded module over a $\mathbb{Z}$-graded ring A and let $d \in \mathbb{Z}$. Recall from Notation 3.32 that $M(d)$ denotes the $\mathbb{Z}$-graded A-module with $M(d)_n = M_{n+d}$.

Definition 4.53 (Serre twists)**.** Let A be an $\mathbb{N}$-graded ring and let $d \in \mathbb{Z}$. We denote by $\mathcal{O}(d)$ the quasi-coherent module over $\mathrm{Proj}(A)$ associated with the $\mathbb{Z}$-graded A-module $A(d)$ (see Construction 4.51). Given $M \in \mathrm{Mod}_{\mathrm{Proj}(A)}$, the dth *Serre twist* of M is the quasi-coherent module over $\mathrm{Proj}(A)$ given by

$$M(d) = M \otimes \mathcal{O}(d).$$

Remark 4.54. If A is generated by $A_{\leq 1}$, then an R-point $x \in \mathrm{Proj}(A)(R)$ is a quotient line $A_1 \otimes_{A_0} R \twoheadrightarrow L$ satisfying some condition. Unraveling the definition, we see that $\mathcal{O}(d)(x) = L^{\otimes d}$.

Proposition 4.55 (Properties of Serre twists)**.** *Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$.*

(i) *For any $d \in \mathbb{Z}$, $\mathcal{O}(d)$ is a line bundle over* $\mathrm{Proj}(A)$*.*

(ii) *For any $\mathbb{Z}$-graded A-module M and $d \in \mathbb{Z}$, there is an isomorphism $\tilde{M}(d) \xrightarrow{\sim} \widetilde{M(d)}$. In particular, for all $d, e \in \mathbb{Z}$, there are isomorphisms*

$$\mathcal{O}(d) \otimes \mathcal{O}(e) \xrightarrow{\sim} \mathcal{O}(d+e) \quad \text{and} \quad \mathcal{O}(1)^{\otimes d} \xrightarrow{\sim} \mathcal{O}(d),$$

so that $\bigoplus_{d\in\mathbb{Z}} \mathcal{O}(d)$ has a structure of quasi-coherent $\mathbb{Z}$-graded algebra over $\mathrm{Proj}(A)$*.*

(iii) *There is a canonical map of $\mathbb{Z}$-graded rings*

$$\mathrm{L}_{A_1} A \to \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathrm{Proj}(A), \mathcal{O}(d)),$$

and, for any $\mathbb{Z}$-graded A-module M, a canonical map of $\mathbb{Z}$-graded A-modules

$$\mathrm{L}_{A_1} M \to \bigoplus_{d\in\mathbb{Z}} \Gamma(\mathrm{Proj}(A), \tilde{M}(d)),$$

which are isomorphisms if A is finitely generated as an A_0-algebra.

Remark 4.56. If A is an arbitrary $\mathbb{N}$-graded ring and M a $\mathbb{Z}$-graded A-module, there are still canonical maps

$$\tilde{M}(d) \to \widetilde{M(d)} \quad \text{and} \quad (\mathrm{L}_{A_+} M)_0 \to \Gamma(\mathrm{Proj}(A), \tilde{M})$$

but they need not be isomorphisms. If A is generated over A_0 by homogeneous elements whose degrees divide d, then $\mathcal{O}(d)$ is a line bundle over $\mathrm{Proj}(A)$ and the first map is an isomorphism.

Definition 4.57 (Tautological line bundle)**.** Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$. The line bundle $\mathcal{O}(1)$ over $\mathrm{Proj}(A)$ is called the *tautological line bundle.*

Example 4.58. If $A = k[x_0, \dots, x_n]$, then

$$\mathrm{L}_{A_1} A = \begin{cases} k[x_0^{\pm 1}], & \text{if } n = 0, \\ A, & \text{if } n \geq 1 \end{cases}$$

(this is exactly the computation of $\mathcal{O}(\mathbb{A}^{n+1}_k - 0)$, see Remark 2.75). Hence, $\Gamma(\mathbb{P}^n_k, \mathcal{O}(d)) \simeq k[x_0, \dots, x_n]_d$ for all $d \geq 0$ and $\Gamma(\mathbb{P}^n_k, \mathcal{O}(d)) = 0$ for all $d < 0$ and $n \geq 1$. For $d = 0$, this says that $\mathcal{O}(\mathbb{P}^n_k) = k$, i.e., that every function on projective n-space is constant.

Example 4.59. Under the description of $\mathrm{Mod}_{\mathbb{P}^1_k}$ from Example 4.46, the quasi-coherent module $\mathcal{O}(d)$ over $\mathbb{P}^1_k$ is the triple $(k[u], k[v], \alpha)$, where $\alpha\colon k[u^{\pm 1}] \xrightarrow{\sim} k[v^{\pm 1}]$ is the $k[t^{\pm 1}]$-linear isomorphism sending 1 to v^d (hence u^n to v^{d-n}).

Example 4.60 (Tautological vector bundles over Grassmannians)**.** Let k be a ring, M a k-module, and $n \in \mathbb{N}$. The *tautological vector bundle* $\mathcal{T}$ on $\mathrm{Gr}_n(M)$ is the rank n vector bundle such that, for any R-point $x\colon \mathrm{Spec}(R) \to \mathrm{Gr}_n(M)$ classifying a quotient space $M \otimes_k R \twoheadrightarrow V$, $\mathcal{T}(x) = V$. For $n = 1$, this recovers the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}(M) = \mathrm{Proj}(\mathrm{Sym}_k(M))$. If $\varpi\colon \mathrm{Gr}_n(M) \hookrightarrow \mathbb{P}(\Lambda^n_k(M))$ is the Plücker embedding (Example 3.100), then $\varpi^*(\mathcal{O}(1)) \simeq \Lambda^n(\mathcal{T})$ in $\mathrm{Vect}_{\mathrm{Gr}_n(M)}$.

Remark 4.61. Let A be an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$. If $f \in A$ is homogeneous of degree d, it defines by Proposition 4.55 a global section of $\mathcal{O}(d)$, or equivalently a map $\mathcal{O}(-d) \to \mathcal{O}$ in $\mathrm{Mod}_{\mathrm{Proj}(A)}$. Hence, any homogeneous subset $F \subset A$ induces a map $\bigoplus_{f\in F} \mathcal{O}(-d_f) \to \mathcal{O}$, where f has degree d_f. One can check that the loci $\mathrm{V}(F)$ and $\mathrm{D}(F)$ in $\mathrm{Proj}(A)$ defined in Remark 3.65 coincide with the vanishing and nonvanishing loci of this map, in the sense of Remark 4.28. This is analogous to Remark 3.45 in the affine case. Thus, the loci associated with maps of quasi-coherent modules unify the vanishing and nonvanishing loci in affine and projective geometry.

Remark 4.62 (Proj and line bundles)**.** Note that the tautological line bundle $\mathcal{O}(1)$ on $\mathrm{Proj}(A)$ depends on the $\mathbb{N}$-graded ring A and not just on the algebraic functor $\mathrm{Proj}(A)$. For example, under the isomorphism $\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(d)})$ for $d \geq 1$, the line bundle $\mathcal{O}(1)$ on $\mathrm{Proj}(A^{(d)})$ corresponds to the line bundle $\mathcal{O}(d)$ on $\mathrm{Proj}(A)$. In fact, by Proposition 4.55(iii), the data of this line bundle is exactly what allows us to recover (the A_1-localization of) A from the algebraic functor $\mathrm{Proj}(A)$.

We can make this more precise as follows. Let $\mathrm{CAlg}^{\mathbb{N},1} \subset \mathrm{CAlg}^{\mathbb{N},\mathrm{es}}$ be the full subcategory spanned by the $\mathbb{N}$-graded rings generated in degrees ≤ 1. Let LineBdl be the category of pairs (X, L) where X is an algebraic functor and $L \in \mathrm{Line}_X$, whose morphisms $(X', L') \to (X, L)$ are pairs (f, λ) with $f \colon X' \to X$ and $\lambda \colon L' \xrightarrow{\sim} f^*(L)$. Then there is functor

$$(\mathrm{Proj}(-), \mathcal{O}(1)) \colon (\mathrm{CAlg}^{\mathbb{N},1})^{\mathrm{op}} \to \mathrm{LineBdl},$$

which is fully faithful on the subcategory of $\mathbb{N}$-graded rings of the form $(\mathrm{L}_{A_1} A)_{\geq 0}$, where A is finitely generated over A_0.

5. Locales and topological spaces

5.1. **Pointless topology.** The starting point of "pointless topology" is the observation that most topological spaces (including for example all Hausdorff spaces) are determined by their posets of open subsets. Since posets are ubiquitous in mathematics, this leads to the appearance of topology in unexpected places. In this chapter, we will see that the poset of open subfunctors of any algebraic functor X is isomorphic to the poset of open subsets of an associated topological space $|X|$, which we will describe explicitly when $X = \mathrm{Spec}(A)$ or $X = \mathrm{Proj}(A)$.

Notation 5.1. Let P be a poset. Given a family $(x_i)_{i \in I}$ in P, we denote by $\bigvee_{i \in I} x_i$ its supremum (least upper bound) and by $\bigwedge_{i \in I} x_i$ its infimum (greatest lower bound), if they exist.

Remark 5.2.

(i) If we view a poset as a category, suprema are colimits and infima are limits.
(ii) The supremum of the empty family (i.e., the colimit of the empty diagram) is the smallest element (i.e., the initial object). Dually, the infimum of the empty family is the largest element.
(iii) A poset admits all suprema if and only if it admits all infima. Indeed, the infimum of a family is the supremum of all elements below the family.

Definition 5.3 (Locale)**.** A *locale* is a poset O satisfying the following conditions:

(i) (Completeness) O admits all suprema (hence all infima).
(ii) (Distributivity) For any $u \in O$ and any family $(v_i)_{i \in I}$,

$$u \wedge \left(\bigvee_{i \in I} v_i \right) = \bigvee_{i \in I} (u \wedge v_i).$$

A morphism of locales $f \colon O' \to O$ is a morphism of posets $f^* \colon O \to O'$ (in the other direction!) that preserves colimits and finite limits. We denote by Loc the category of locales.

Example 5.4 (Totally ordered sets)**.** Any totally ordered set with suprema is a locale. For example, the posets $\mathbb{N} \cup \{+\infty\}$ and $\mathbb{R} \cup \{\pm\infty\}$ are locales.

Example 5.5 (The locale of a topological space)**.** Let T be a topological space. The poset $\mathrm{Open}(T)$ of open subsets of T is a locale:

- Given a family of open subsets $(U_i)_{i \in I}$, its supremum is the union $\bigcup_{i \in I} U_i$ and its infimum is the interior of the intersection $\bigcap_{i \in I} U_i$.
- The distributivity law follows from the set-theoretic distributive law and the fact that *finite* intersections of open subsets are open.

If $f \colon T \to S$ is a map of topological spaces, then $f^{-1} \colon \mathrm{Open}(S) \to \mathrm{Open}(T)$ is a morphism of posets that preserves colimits and finite limits. Hence, it is a morphism of locales $\mathrm{Open}(T) \to \mathrm{Open}(S)$. This defines a functor

$$\mathrm{Open} \colon \mathrm{Top} \to \mathrm{Loc}.$$

Remark 5.6. Locales are also called *frames* or *complete Heyting algebras*. However, these other names come with different notions of morphisms, hence describe different categories.

Proposition 5.7 (Limits and colimits of locales)**.** *The category* Loc *admits limits and colimits, and the inclusion* $\mathrm{Loc} \hookrightarrow \mathrm{Pos}^{\mathrm{op}}$ *preserves colimits. The initial object of* Loc *is* $0 = \mathrm{Open}(\varnothing) = \{\varnothing\}$ *and the final object is* $1 = \mathrm{Open}(*) = \{\varnothing \to *\}$.

Definition 5.8 (Points of a locale)**.** Let O be a locale. We define a topological space $\mathrm{Pt}(O)$ as follows:

- The elements of $\mathrm{Pt}(O)$ are the *points* of O, i.e., the morphisms of locales $1 \to O$.
- The open sets of $\mathrm{Pt}(O)$ are the subsets $\mathrm{Pt}(u) = \{p\colon 1 \to O \mid p^*(u) = *\}$ for all $u \in O$.

Given a morphism of locales $f\colon O' \to O$, the induced map $\mathrm{Pt}(O') \to \mathrm{Pt}(O)$ is continuous, since the preimage of $\mathrm{Pt}(u)$ is $\mathrm{Pt}(f^*(u))$. Hence, we have a functor

$$\mathrm{Pt}\colon \mathrm{Loc} \to \mathrm{Top}.$$

Remark 5.9 (Explicit description of points)**.** Let P be a poset. A map of *sets* $f\colon P \to \{\varnothing \to *\}$ is determined by the subset $F = f^{-1}(*) \subset P$. It is a map of posets if and only if:

(i) F is upward-closed.

If this is the case and P has finite limits, then f preserves finite limits if and only if:

(ii) F is nonempty and $x \wedge y \in F$ whenever $x, y \in F$.

If moreover P is complete, then f preserves colimits if and only if:

(iii) whenever $\bigvee_{i\in I} x_i \in F$, there exists $i \in I$ with $x_i \in F$.

Subsets $F \subset P$ satisfying (i) and (ii) are called *filters* on P, and they are called *completely prime* if they also satisfy (iii). Thus, we can identify the points of a locale O with the completely prime filters on O.

Proposition 5.10 (The adjunction between topological spaces and locales)**.** *The functor* $\mathrm{Open}\colon \mathrm{Top} \to \mathrm{Loc}$ *is left adjoint to the functor* $\mathrm{Pt}\colon \mathrm{Loc} \to \mathrm{Top}$. *This adjunction is idempotent, i.e., it restricts to an equivalence between the essential images of both functors.*

Definition 5.11 (Sober spaces and spatial locales)**.**

(i) A topological space T is *sober* if it lies in the essential image of $\mathrm{Pt}\colon \mathrm{Loc} \to \mathrm{Top}$, or equivalently if the unit map $T \to \mathrm{Pt}(\mathrm{Open}(T))$ is an isomorphism.
(ii) A locale O is *spatial* if it lies in the essential image of $\mathrm{Open}\colon \mathrm{Top} \to \mathrm{Loc}$, or equivalently if the counit map $\mathrm{Open}(\mathrm{Pt}(O)) \to O$ is an isomorphism.

Remark 5.12. By Proposition 5.10, the adjunction $\mathrm{Open} \dashv \mathrm{Pt}$ restricts to an equivalence between the category $\mathrm{Top}^{\mathrm{sob}}$ of sober topological spaces and the category $\mathrm{Loc}^{\mathrm{spa}}$ of spatial locales. Moreover, the inclusion $\mathrm{Top}^{\mathrm{sob}} \hookrightarrow \mathrm{Top}$ has a left adjoint given by $\mathrm{Pt} \circ \mathrm{Open}$, called *soberification*, and the inclusion $\mathrm{Loc}^{\mathrm{spa}} \hookrightarrow \mathrm{Loc}$ has a right adjoint given by $\mathrm{Open} \circ \mathrm{Pt}$, called *spatialization*. In particular, limits of sober spaces are sober and colimits of spatial locales are spatial.

Proposition 5.13 (Characterization of sober spaces)**.** *Let T be a topological space. Then the points of the locale* $\mathrm{Open}(T)$ *can be identified with the irreducible closed subsets of T, and the unit map* $T \to \mathrm{Pt}(\mathrm{Open}(T))$ *sends t to the closure of $\{t\}$. Hence, a topological space T is sober if and only if the map*

$$\{\textit{points of } T\} \to \{\textit{irreducible closed subsets of } T\},$$
$$t \mapsto \overline{\{t\}},$$

is a bijection, i.e., if and only if every irreducible closed subset of T has a unique generic point.

Remark 5.14.

(i) Every Hausdorff space is sober, as its irreducible subsets are singletons.
(ii) Being sober is a local property: if T admits an open covering by sober spaces, then T is sober.

5.2. **Locales of radical ideals.** Let R be a ring. Recall that the poset of open subfunctors of $\mathrm{Spec}(R)$ is isomorphic to the poset Rad_R of radical ideals in R (Proposition 2.66(ii)). We claim that it is a locale. The supremum of a family of radical ideals $(K_i)_{i\in I}$ is the radical of $\sum_{i\in I} K_i$. The distributivity law reads

$$K \cap \sqrt{\sum_{i\in I} L_i} = \sqrt{\sum_{i\in I} (K \cap L_i)},$$

and it follows from the following facts: we have $K \cap L = \sqrt{KL}$ for any radical ideals K and L, and the *product* of ideals distributes over sums. By contrast, the poset Id_R of all ideals is usually not a locale, since the intersection of ideals does not distribute over sums.

For any ring map $R \to S$, the base change map $\mathrm{Id}_R \to \mathrm{Id}_S$, $I \mapsto IS$, preserves products and sums of ideals. It follows that the base change map $\mathrm{Rad}_R \to \mathrm{Rad}_S$, $I \mapsto \sqrt{IS}$, preserves finite intersections and suprema, hence is a morphism of locales $\mathrm{Rad}_S \to \mathrm{Rad}_R$.

We now compute the points of the locale Rad_R.

Definition 5.15 (Prime spectrum)**.** Let R be a ring. The *prime spectrum* of R is the topological space $\mathrm{Prim}(R)$ defined as follows:

- The elements of $\mathrm{Prim}(R)$ are the prime ideals of R.
- The open sets of $\mathrm{Prim}(R)$ are the subsets $\mathrm{Prim}(I) = \{\mathfrak{p} \mid I \not\subset \mathfrak{p}\}$ for all $I \subset R$.

Since prime ideals are radical, $\mathrm{Prim}(I)$ depends only on the radical ideal $\sqrt{(I)}$ generated by I.

Proposition 5.16 (The locale of a ring)**.** *Let R be a ring.*

(i) *There is a homeomorphism*
$$\mathrm{Prim}(R) \xrightarrow{\sim} \mathrm{Pt}(\mathrm{Rad}_R), \quad \mathfrak{p} \mapsto \{I \in \mathrm{Rad}_R \mid I \not\subset \mathfrak{p}\},$$
under which the open subset $\mathrm{Pt}(I)$ *corresponds to the open subset* $\mathrm{Prim}(I)$*.*

(ii) *The locale* Rad_R *is spatial, i.e., the map from* (i) *induces an isomorphism* $\mathrm{Open}(\mathrm{Prim}(R)) \xrightarrow{\sim} \mathrm{Rad}_R$*.*

(iii) *The topological space* $\mathrm{Prim}(R)$ *is* spectral*, i.e., it is sober and its quasi-compact open subsets form a basis of the topology that is closed under finite intersections.*

Remark 5.17. One can show that, conversely, every spectral space is homeomorphic to $\mathrm{Prim}(R)$ for some ring R (which is however far from unique).

Let now A be an $\mathbb{N}$-graded ring, and let hRad_A be the poset of saturated radical homogeneous ideals in A. By Proposition 3.81 (and Remark 3.82), the poset hRad_A is isomorphic to the poset of open subfunctors of $\mathrm{Proj}(A)$. This poset is also a locale: the distributivity law follows from the one in Rad_A and the fact that saturation preserves finite intersections. We now compute the points of hRad_A.

Definition 5.18 (Homogeneous prime spectrum)**.** Let A be an $\mathbb{N}$-graded ring. The *homogeneous prime spectrum* of A is the topological space $\mathrm{hPrim}(A)$ defined as follows:

- The elements of $\mathrm{hPrim}(A)$ are the saturated homogeneous prime ideals of A, or equivalently the homogeneous prime ideals of A that do not contain A_+ (Remark 3.72(iv)).
- The open sets of $\mathrm{hPrim}(R)$ are the subsets $\mathrm{hPrim}(I) = \{\mathfrak{p} \mid I \not\subset \mathfrak{p}\}$ for all homogeneous subsets $I \subset A$.

Note that $\mathrm{hPrim}(I)$ depends only on the saturated radical ideal $\sqrt{(I)}^{\mathrm{sat}}$ generated by I.

Proposition 5.19 (The locale of an $\mathbb{N}$-graded ring)**.** *Let A be an $\mathbb{N}$-graded ring.*

(i) *There is a homeomorphism*
$$\mathrm{hPrim}(A) \xrightarrow{\sim} \mathrm{Pt}(\mathrm{hRad}_A), \quad \mathfrak{p} \mapsto \{I \in \mathrm{hRad}_A \mid I \not\subset \mathfrak{p}\},$$
under which the open subset $\mathrm{Pt}(I)$ *corresponds to the open subset* $\mathrm{hPrim}(I)$*.*

(ii) *The locale* hRad_A *is spatial, i.e., the map from* (i) *induces an isomorphism* $\mathrm{Open}(\mathrm{hPrim}(A)) \xrightarrow{\sim} \mathrm{hRad}_A$*.*

(iii) *The topological space* $\mathrm{hPrim}(A)$ *is sober and its quasi-compact open subsets form a basis of the topology that is closed under* binary *intersections. If A is finitely generated as an A_0-algebra, then* $\mathrm{hPrim}(A)$ *is spectral.*

5.3. **The topological space of an algebraic functor.** Let $\mathrm{Open}(X)$ denote the poset of open subfunctors of an algebraic functor X. Since the preimage of an open subfunctor is open, we have a functor
$$\mathrm{Open}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Pos}^{\mathrm{op}}.$$
By Proposition 4.23(ii), $\mathrm{Open}(X)$ is isomorphic to the poset Rad_X of quasi-coherent radical ideals over X, so that there is an isomorphism of posets
$$\mathrm{Open}(X) \xrightarrow{\sim} \lim_{\mathrm{Spec}(R)\to X} \mathrm{Open}(\mathrm{Spec}(R)).$$
In §5.2, we saw that the restriction of Open to the subcategory $\mathrm{Aff} \subset \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$ of affine schemes lands in the subcategory of locales. Combining these facts with Proposition 5.7, we immediately deduce the following result:

Proposition 5.20. *The functor* $\mathrm{Open}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Pos}^{\mathrm{op}}$ *lands in the subcategory* $\mathrm{Loc} \subset \mathrm{Pos}^{\mathrm{op}}$*, and the induced functor*
$$\mathrm{Open}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Loc}$$
is colimit-preserving.

Since colimits of spatial locales are spatial (Remark 5.12), we deduce:

Corollary 5.21. *For any algebraic functor X, the locale* $\mathrm{Open}(X)$ *is spatial.*

Remark 5.22. The final object in $\mathrm{Open}(X)$ is $\mathrm{D}(1) = X$ and the initial object is $\mathrm{D}(0) = \varnothing_X$, where $\varnothing_X(0) = X(0)$ and $\varnothing_X(R) = \varnothing$ for all $R \neq 0$. For any $U, V \in \mathrm{Open}(X)$, we have $U \wedge V = U \cap V$.

We now define an explicit topological space $|X|$ whose locale of open sets is $\mathrm{Open}(X)$. If $X = \mathrm{Spec}(R)$, the prime spectrum $\mathrm{Prim}(R)$ is the unique sober space with this property, by Proposition 5.16. On the other hand, by Remark 3.4, there is a bijection

$$\{\text{connected components of } \mathrm{Field}_R\} \xrightarrow{\sim} \mathrm{Prim}(R), \quad (\varphi\colon R \to k) \mapsto \ker(\varphi),$$

with inverse given by $\mathfrak{p} \mapsto \kappa(\mathfrak{p})$.

Construction 5.23 (Underlying space of an algebraic functor)**.** Let X be an algebraic functor and let $\mathrm{Field}_X \subset \mathrm{El}(X)^{\mathrm{op}}$ be the full subcategory spanned by the field-valued points of X. We define a topological space $|X|$ as follows:

- The points of $|X|$ are the connected components of the category Field_X.
- A subset $U \subset |X|$ is open if and only if, for every ring R and every R-point $x\colon \mathrm{Spec}(R) \to X$, the preimage $|x|^{-1}(U) \subset \mathrm{Prim}(R)$ is open.

Proposition 5.24. *The functor*

$$\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Top}, \quad X \mapsto |X|,$$

is colimit-preserving. Hence, for any algebraic functor X, there is a natural isomorphism of posets

$$\mathrm{Open}(X) \xrightarrow{\sim} \mathrm{Open}(|X|), \quad U \mapsto |U|.$$

Remark 5.25.

(i) Since every field admits an algebraic closure, every point $x \in |X|$ is represented by a k-point where k is an algebraically closed field; such a k-point is sometimes called a *geometric point* of X, although the exact meaning of "geometric point" varies from sources to sources.

(ii) If X is affine, then $|X|$ is sober. However, as colimits of sober spaces need not be sober, $|X|$ is not sober in general. Its soberification is the space $\mathrm{Pt}(\mathrm{Open}(X))$.

(iii) If $X = \mathrm{Proj}(A)$, then $|X|$ is sober as it is covered by the open subspaces $|\mathrm{Spec}(A_{(f)})|$, which are sober. Hence, the canonical map $|X| \to \mathrm{Pt}(\mathrm{Open}(X)) \simeq \mathrm{hPrim}(A)$ is a homeomorphism.

Remark 5.26 (Points in algebraic geometry)**.** We now have introduced two distinct notions of "points" of an algebraic functor X:

(i) If R is any ring, an *R-point* or *R-valued point* of X is an element of $X(R)$.

(ii) A *point* of X can also refer to a point of the topological space $|X|$, which is an equivalence class of field-valued points of X.

For fixed R, the R-points of X only form a set $X(R)$, but for varying R they form a category, namely the category of elements $\mathrm{El}(X)$. On the other hand, the points of X in the sense of (ii) form a topological space $|X|$. These two notions of points are related by the zigzag of functors

$$\mathrm{El}(X)^{\mathrm{op}} \hookleftarrow \mathrm{Field}_X \twoheadrightarrow |X|.$$

Lemma 5.27 (Underlying space of pullbacks)**.** *Let $Y \to X \leftarrow Z$ be maps of algebraic functors. Then the canonical map*

$$|Y \times_X Z| \to |Y| \times_{|X|} |Z|$$

is surjective. It is bijective if $Y \to X$ is a monomorphism.

Proposition 5.28 (Underlying space of immersions)**.** *Let X be an algebraic functor and let $Y \hookrightarrow X$ be an open immersion, a closed immersion, or an immersion. Then the induced map $|Y| \to |X|$ is an open embedding, a closed embedding, or a locally closed embedding, respectively.*

Remark 5.29 (Closed subfunctors vs. closed subspaces)**.** By Proposition 5.28, there is a map of posets

$$\mathrm{Closed}(X) \to \mathrm{Closed}(|X|), \quad Z \mapsto |Z|.$$

In contrast to Proposition 5.24, this map is almost never bijective, but it is surjective if $X = \mathrm{Spec}(A)$ or if $X = \mathrm{Proj}(A)$: a preimage of a closed subset of $|X|$ is given by the vanishing locus of the radical ideal in A corresponding to the open complement. In general, if $Z \subset X$ is a closed subfunctor with open complement U (Definition 4.24), then $|Z| = |X| - |U|$.

Definition 5.30 (Topological properties of algebraic functors)**.** Let P be a property of topological spaces, such as *connected*, *locally connected*, *irreducible*, or *discrete*. We say that an algebraic functor X has property P if $|X|$ has property P.[5]

Example 5.31. If $X = \mathrm{Spec}(R)$, so that $|X|$ is the prime spectrum $\mathrm{Prim}(R)$, we have the following standard results from commutative algebra:

(i) X is connected if and only if R has exactly two idempotent elements (see Remark 5.43 below for a generalization of this statement to arbitrary algebraic functors).
(ii) X is irreducible if and only if the nilradical $\sqrt{0} \subset R$ is prime.
(iii) X is discrete if and only if R is artinian.

Definition 5.32 (Topological properties of maps of algebraic functors)**.** Let P be a property of maps of topological spaces, such as *surjective*, *injective*, *bijective*, *dominant*, *submersive*, *open*, *closed*, or *homeomorphism*.[6] A map of algebraic functors $f\colon Y \to X$ is said to have property P if the map $|f|\colon |Y| \to |X|$ has property P. We say that f is *universally* P if for every $X' \to X$, the base change $Y \times_X X' \to X'$ has property P.

Proposition 5.33. *Let $f\colon Y \to X$ be a map of algebraic functors.*

(i) *Let P be any of the following properties: surjective, injective, bijective, dominant, submersive, open, closed, or homeomorphism. Then f is universally P if and only if, for every ring R and every $\mathrm{Spec}(R) \to X$, the base change $Y \times_X \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ has property P.*
(ii) *f is universally surjective if and only if it is surjective.*
(iii) *If f is a monomorphism, then it is universally injective.*
(iv) *If f is a Zariski-local epimorphism, then it is universally submersive and in particular universally surjective.*

Example 5.34. With the exception of surjectivity, the properties listed in Proposition 5.33(i) are not automatically universal, even for maps between affine schemes:

(i) The map $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ is a homeomorphism, but it is not universally injective: its pullback along itself is $\mathrm{Spec}(\mathbb{C} \times \mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$, which induces the map of topological spaces $* \sqcup * \to *$.
(ii) If k is a field, then $|\mathrm{Spec}(k)| = *$ and hence any map $X \to \mathrm{Spec}(k)$ is closed. The map $\mathbb{A}^1_k \to \mathrm{Spec}(k)$ is however not universally closed: its pullback along itself is the projection $\mathbb{A}^2_k \to \mathbb{A}^1_k$, which is not closed as it restricts to an isomorphism $\mathrm{V}(xy-1) \xrightarrow{\sim} \mathbb{A}^1_k - 0$.
(iii) An example of an open morphism that is not universally open is

$$f\colon\ \mathrm{Spec}(k[t]) \to \mathrm{Spec}(k[x,y]/(y^2 - x^3 - x^2)), \quad x \mapsto t^2 - 1, \quad y \mapsto t(t^2-1),$$

where k is a field of characteristic not 2, which is the resolution of singularities of the affine nodal cubic. The map $|f|$ is the quotient map $|\mathbb{A}^1_k| \twoheadrightarrow |\mathbb{A}^1_k|/(-1 \sim 1)$, which is open but whose pullback along itself is not open. By Lemma 5.27, this implies that the pullback of f along itself is not open.

Corollary 5.35 (Zariski codescent for the underlying space)**.** *Let $(Y_i \to X)_{i\in I}$ be a family of maps of algebraic functors such that $\coprod_{i\in I} Y_i \to X$ is a Zariski-local epimorphism. Then the diagram*

$$\coprod_{i,j\in I} |Y_i \times_X Y_j| \rightrightarrows \coprod_{i\in I} |Y_i| \to |X|$$

is a coequalizer diagram of topological spaces.

Definition 5.36 (Residue field)**.** Let X be an algebraic functor and let $x \in |X|$. Let $\mathrm{Field}_{(X,x)}$ be the full subcategory of Field_X spanned by the field-valued points hitting x. If the category $\mathrm{Field}_{(X,x)}$ has an initial object, we call it the *residue field* of X at x and denote it by $\kappa(x)$.

Let $f\colon Y \to X$ be a map of algebraic functors, let $y \in |Y|$ and let $x = f(y) \in |X|$. If both residue fields $\kappa(y)$ and $\kappa(x)$ exist, then the universal property of $\kappa(x)$ gives a map of fields $\kappa(x) \to \kappa(y)$, called the *residual field extension* of f at y.

[5] Provided this does not conflict with any other definitions. For example, an affine scheme X is called *noetherian* if the ring $\mathcal{O}(X)$ is noetherian, which is strictly stronger than $|X|$ being noetherian; the latter property is then called *topologically noetherian*.

[6] A continuous map $f\colon T \to S$ is *dominant* if $f(T)$ is dense in S, and it is *submersive* if it is surjective and S has the quotient topology.

Proposition 5.37 (Existence of residue fields)**.** *Let X be an algebraic functor, let $Y \hookrightarrow X$ be an immersion, and let $y \in |Y|$ be a point with image $x \in |X|$. Then the functor* $\mathrm{Field}_{(Y,y)} \to \mathrm{Field}_{(X,x)}$ *is an equivalence. In particular, the residue field of X at x exists if and only if the residue field of Y at y exists, in which case the residual field extension $\kappa(x) \to \kappa(y)$ is an isomorphism.*

Example 5.38.

(i) Let $X = \mathrm{Spec}(R)$ and let $x \in |X|$ be given by the prime ideal $\mathfrak{p} \subset R$. Then the residue field $\kappa(x)$ exists and is the usual residue field $\kappa(\mathfrak{p})$, by Remark 3.4.

(ii) Let $X = \mathrm{Proj}(A)$ and let $x \in |X|$ be given by the saturated homogeneous prime ideal $\mathfrak{p} \subset A$. By Remark 3.72(iv), there exists a homogeneous element $f \in A_+$ with $f \notin \mathfrak{p}$. Then $\mathfrak{p}_{(f)}$ is a prime ideal in $A_{(f)}$ and the open immersion $\mathrm{Spec}(A_{(f)}) \hookrightarrow \mathrm{Proj}(A)$ sends $\mathfrak{p}_{(f)}$ to $\mathfrak{p}$. By Proposition 5.37 and (i), the residue field $\kappa(x)$ exists and is given by $\kappa(\mathfrak{p}_{(f)}) \simeq \kappa(\mathfrak{p})_0$.

5.4. **Open coverings.** The notion of *open covering* of a topological space T depends only on the locale $\mathrm{Open}(T)$: a family of open subsets $(U_i \subset T)_{i\in I}$ is an open covering if and only if its supremum in $\mathrm{Open}(T)$ equals T. In the locale of an algebraic functor, there are several useful characterizations of this property:

Proposition 5.39. *Let X be an algebraic functor, let $(U_i \subset X)_{i\in I}$ be a family of open subfunctors, and let $K_i \in \mathrm{Rad}_X$ be the quasi-coherent radical ideal of U_i. The following conditions are equivalent:*

(i) $X = \bigvee_{i\in I} U_i$ *in the poset* $\mathrm{Open}(X)$.
(ii) $\mathcal{O}_X = \bigvee_{i\in I} K_i$ *in the poset* Rad_X.
(iii) *For every ring R and every $x\colon \mathrm{Spec}(R) \to X$, $\bigcup_{i\in I} K_i(x)$ generates the unit ideal in R.*
(iv) *For every local ring R, $X(R) = \bigcup_{i\in I} U_i(R)$.*
(v) *For every field k, $X(k) = \bigcup_{i\in I} U_i(k)$.*
(vi) *The induced map $\coprod_{i\in I} U_i \to X$ is a Zariski-local epimorphism (Definition 4.39).*
(vii) *The induced map $\coprod_{i\in I} U_i \to X$ is surjective (Definition 5.32).*

Definition 5.40 (Open covering)**.** A family of open subfunctors $(U_i \subset X)_{i\in I}$ is called an *open covering* of X if it satisfies the equivalent conditions of Proposition 5.39.

Remark 5.41. By Proposition 5.39(vii), open coverings of an algebraic functor X are equivalent to open coverings of its underlying space $|X|$.

Example 5.42 (Open coverings of affine and projective schemes)**.**

(i) Let R be a ring and $(F_i)_{i\in I}$ a family of subsets of R. Then $(\mathrm{D}(F_i) \subset \mathrm{Spec}(R))_{i\in I}$ is an open covering of $\mathrm{Spec}(R)$ if and only if $\bigcup_{i\in I} F_i$ generates the unit ideal of R, by Proposition 2.66(ii). In general, the subfunctors $\mathrm{D}(F_i)$ form an open covering of $\mathrm{D}\big(\bigcup_{i\in I} F_i\big)$.

(ii) Let R be a ring, L a line over R, and $(s_i)_{i\in I}$ a family of elements of L. Then $(\mathrm{D}(s_i) \subset \mathrm{Spec}(R))_{i\in I}$ is an open covering of $\mathrm{Spec}(R)$ if and only if $(s_i)_{i\in I}$ generates L.

(iii) Let A be an $\mathbb{N}$-graded ring and $(F_i)_{i\in I}$ a family of homogeneous subsets of R. Then $(\mathrm{D}(F_i) \subset \mathrm{Proj}(A))_{i\in I}$ is an open covering of $\mathrm{Proj}(A)$ if and only if the saturated radical ideal generated by $\bigcup_{i\in I} F_i$ is the unit ideal of A, by Proposition 3.81(ii). For example, the affine open subschemes $\mathrm{D}(f) \simeq \mathrm{Spec}(A_{(f)})$ with $f \in A_+$ homogeneous form an open covering of $\mathrm{Proj}(A)$. In general, the subfunctors $\mathrm{D}(F_i)$ form an open covering of $\mathrm{D}\big(\bigcup_{i\in I} F_i\big)$.

(iv) Let k be a ring, M a k-module, and $n \in \mathbb{N}$. For every $\alpha \in M^n$, let $U(\alpha) \subset \mathrm{Gr}_n(M)$ be the subfunctor consisting of the quotient spaces $\varphi\colon M \otimes_k R \twoheadrightarrow V$ such that $\varphi \circ \alpha\colon R^n \to V$ is an isomorphism, which is open by Proposition 3.47(iv). If $(\alpha_i)_{i\in I}$ is a family in M^n, then $(U(\alpha_i) \subset \mathrm{Gr}_n(M))_{i\in I}$ is an open covering of $\mathrm{Gr}_n(M)$ if and only if the image of $(\alpha_i)_{i\in I}$ in $\Lambda^n_k M$ is a generating family (cf. Example 3.100).

Remark 5.43 (Clopen subfunctors vs. clopen subspaces)**.** For any algebraic functor X, the map

$$\mathrm{Clopen}(X) \to \mathrm{Clopen}(|X|), \quad U \mapsto |U|,$$

is bijective. Indeed, let $U \subset X$ be an open subfunctor such that $|U|$ is clopen in $|X|$, and let $V \subset X$ be the open subfunctor such that $|X| = |U| \sqcup |V|$. Then U and V form an open covering of X such that $U \cap V = \varnothing_X$, so that the map $\mathcal{O}(X) \to \mathcal{O}(U) \times \mathcal{O}(V)$ is an isomorphism by Corollary 4.43. We therefore find an idempotent function $e \in \mathcal{O}(X)$ such that $U \subset \mathrm{D}(e)$ and $V \subset \mathrm{D}(1-e)$. Since $U \vee V = X$ and $\mathrm{D}(e) \wedge \mathrm{D}(1-e) = \varnothing_X$ in $\mathrm{Open}(X)$, the distributivity law implies that $U = \mathrm{D}(e)$. As e is idempotent, we have $\mathrm{D}(e) = \mathrm{V}(1-e)$, so that U is closed, as desired. In particular, by Proposition 4.29, X is connected (in the sense of Definition 5.30) if and only if the ring $\mathcal{O}(X)$ has exactly two idempotent elements.

Proposition 5.44 (Closure properties of open coverings)**.**

(i) (Refinement) *A family of open subfunctors* $(U_i \subset X)_{i\in I}$ *is an open covering if it is refined by an open covering* $(V_j \subset X)_{j\in J}$*, i.e., if there exists* $\alpha\colon J \to I$ *such that* $V_j \subset U_{\alpha(j)}$ *for all* $j \in J$*.*

(ii) (Composition) *If* $(U_i \subset X)_{i\in I}$ *and* $(V_{ij} \subset U_i)_{j\in J_i}$ *are open coverings, then* $(V_{ij} \subset X)_{(i,j)\in\coprod_{i\in I} J_i}$ *is an open covering.*

(iii) (Intersection) *If* $(U_i \subset X)_{i\in I}$ *and* $(V_j \subset X)_{j\in J}$ *are open coverings, then* $(U_i \cap V_j \subset X)_{(i,j)\in I\times J}$ *is an open covering.*

(iv) (Base change) *If* $(U_i \subset X)_{i\in I}$ *is an open covering and* $f\colon Y \to X$ *is a map of algebraic functors, then* $(f^{-1}(U_i) \subset Y)_{i\in I}$ *is an open covering.*

6. Sheaves

In this chapter, we introduce the formalism of Grothendieck topologies and sheaves. A Grothendieck topology τ on a category $\mathcal{C}$ determines (and is determined by) a subcategory $\mathrm{Sh}_\tau(\mathcal{C})$ of the presheaf category $\mathrm{P}(\mathcal{C})$, whose objects are called *τ-sheaves* on $\mathcal{C}$. These are presheaves on $\mathcal{C}$ satsfying a certain local-to-global condition with respect to the topology τ, called *τ-descent*.

In algebraic geometry, we typically use this formalism in the following situations:

(i) $\mathcal{C}$ is the category of open subsets of a topological space T, which has a canonical Grothendieck topology; one then speaks of *sheaves on T*. This was the original notion of sheaf discovered by Leray in the 1940s, before Grothendieck's categorical generalization in the 1950s.

(ii) $\mathcal{C} = \mathrm{CAlg}_k^{\mathrm{op}}$ is the opposite of the category of k-algebras, or equivalently the category Aff_k of affine k-schemes, on which there is a plethora of commonly used Grothendieck topologies. The main ones are the Zariski, Nisnevich, étale, fppf ("fidèlement plat de présentation finie"), and fpqc ("fidèlement plat quasi-compact") topologies.

(iii) The topologies in (ii) extend to various enlargements of Aff_k, such as the category Sch_k of k-schemes (to be introduced in §7) and even the whole category $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ of algebraic k-functors.

All the "Zariski descent" statements established so far are examples of descent statements for the so-called Zariski topology on $\mathrm{CAlg}^{\mathrm{op}}$ (e.g., Corollary 2.72 and Proposition 3.57) or on $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ (e.g., Corollary 4.43).

6.1. **Sieves and descent.**

Definition 6.1 (Sieve)**.** Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$. A *sieve* on X is a subpresheaf of the representable presheaf $よ(X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$.

Remark 6.2 (Sieves as subcategories)**.** A *left closed* subcategory of a category $\mathcal{C}$ is a full subcategory $\mathcal{D} \subset \mathcal{C}$ such that, if $X \in \mathcal{D}$ and $Y \to X$ is any morphism in $\mathcal{C}$, then $Y \in \mathcal{D}$. If R is a sieve on an object $X \in \mathcal{C}$, its category of element $\mathrm{El}(R)$ is a left closed subcategory of $\mathrm{El}(よ(X)) = \mathcal{C}_{/X}$. This defines an order-preserving bijection

$$\mathrm{El}\colon \{\text{sieves on } X\} \xrightarrow{\sim} \{\text{left closed subcategories of } \mathcal{C}_{/X}\}.$$

In practice, we think of sieves on X either as subfunctors of $よ(X)$ or as left closed subcategories of $\mathcal{C}_{/X}$, depending on the situation.

Notation 6.3 (Pullback of sieves)**.** Let R be a sieve on X and let $f\colon Y \to X$ be a map. We denote by $f^*(R)$ the pullback $R \times_{よ(X)} よ(Y)$, which is a sieve on Y. Concretely, $f^*(R)$ consists of all maps to Y whose composition with f belongs to R.

Definition 6.4 (Generated sieve)**.** Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$. The sieve on X *generated by* a family of maps $(Y_i \to X)_{i\in I}$ is the image of the map $\coprod_{i\in I} よ(Y_i) \to よ(X)$, i.e., the sieve consisting of all maps to X that factor through Y_i for some i.

Definition 6.5 (Descent)**.** Let $\mathcal{C}$ be a category, $F \in \mathrm{P}(\mathcal{C})$ a presheaf on $\mathcal{C}$, and R a sieve on $X \in \mathcal{C}$. We say that F *satisfies descent* along R if the inclusion $R \subset よ(X)$ induces an isomorphism

$$F(X) = \mathrm{Map}(よ(X), F) \xrightarrow{\sim} \mathrm{Map}(R, F).$$

Example 6.6 (The empty sieve)**.** For any $X \in \mathcal{C}$, the empty subpresheaf of $よ(X)$ is a sieve on X, called the *empty sieve* on X. Since the empty presheaf is an initial object in $\mathrm{P}(\mathcal{C})$, a presheaf F satisfies descent along this sieve if and only if $F(X)$ is a one-point set.

Example 6.7 (Sieve generated by two subobjects)**.** Let $X \in \mathcal{C}$, let $U, V \subset X$ be a pair of subobjects of X, and let R be the sieve on X generated by U and V. If the intersection $U \cap V$ exists, then R is the pushout $よ(U) \sqcup_{よ(U\cap V)} よ(V)$ in $\mathrm{P}(\mathcal{C})$. Hence, a presheaf F on $\mathcal{C}$ satisfies descent along R if and only if the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U\cap V) \end{array}$$

is cartesian.

Lemma 6.8 (Presentations of sieves)**.** *Let R be a sieve on X and let $(Y_i \to X)_{i\in I}$ be a family of maps generating R. Then the maps $よ(Y_i) \to R$ induce an isomorphism of presheaves*

$$\operatorname{colim}\left(\coprod_{i,j\in I} よ(Y_i) \times_{よ(X)} よ(Y_j) \rightrightarrows \coprod_{i\in I} よ(Y_i)\right) \xrightarrow{\sim} R.$$

Proposition 6.9 (Characterization of descent)**.** *Let R be a sieve on $X \in \mathcal{C}$ and let $(Y_i \to X)_{i\in I}$ be a family of maps generating R. For a presheaf $F \in \mathrm{P}(\mathcal{C})$, the following conditions are equivalent:*

(i) *F satisfies descent along R.*

(ii) *The canonical map*

$$F(X) \to \lim_{Y\in \mathrm{El}(R)} F(Y)$$

is an isomorphism.

(iii) *The diagram*

$$F(X) \to \prod_{i\in I} F(Y_i) \rightrightarrows \prod_{i,j\in I} \mathrm{Map}(よ(Y_i) \times_{よ(X)} よ(Y_j), F)$$

is an equalizer.

Remark 6.10. If in Proposition 6.9 the pullbacks $Y_i \times_X Y_j$ exist in $\mathcal{C}$, we can rewrite Condition (iii) as follows:

(iii′) The diagram

$$F(X) \to \prod_{i\in I} F(Y_i) \rightrightarrows \prod_{i,j\in I} F(Y_i \times_X Y_j)$$

is an equalizer.

Example 6.11 (Monogenic sieve)**.** Let R be the sieve on $X \in \mathcal{C}$ generated by a single map $Y \to X$. If the pullback $Y \times_X Y$ exists in $\mathcal{C}$, then a presheaf $F \in \mathrm{P}(\mathcal{C})$ satisfies descent along R if and only if the following diagram is an equalizer:

$$F(X) \to F(Y) \rightrightarrows F(Y \times_X Y).$$

Remark 6.12 (Descent for categories)**.** We can define descent for a presheaf of categories $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ (in the sense of Definition 4.1) via Condition (ii) of Proposition 6.9. Namely, if R is a sieve on $X \in \mathcal{C}$, we say that F *satisfies descent* along R if the canonical functor

$$F(X) \to \lim_{Y\in \mathrm{El}(R)} F(Y)$$

is an equivalence of categories. It turns out that this is *not* equivalent to Condition (iii) of Proposition 6.9. Instead, assuming that the relevant fiber products exist in $\mathcal{C}$ for simplicity, it is equivalent to the following condition:

(iii″) The diagram of categories

$$F(X) \to \prod_{i\in I} F(Y_i) \rightrightarrows \prod_{i,j\in I} F(Y_i \times_X Y_j) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod_{i,j,k\in I} F(Y_i \times_X Y_j \times_X Y_k)$$

is a limit diagram.

This explains why the descent condition takes this more complicated form for presheaves of categories. Note that (iii″) is equivalent to (iii′) if F is a presheaf of *posets*.

6.2. Grothendieck topologies and sheaves.

Definition 6.13 (Grothendieck topology, site)**.** A *Grothendieck topology* τ on a category $\mathcal{C}$, or *topology* for short, assigns to each object $X \in \mathcal{C}$ a collection of sieves on X, called *τ-covering sieves*, subject to the following conditions:

(i) If R is a τ-covering sieve on X and $f\colon Y \to X$ is any map, then the pullback $f^*(R)$ is a τ-covering sieve on Y.
(ii) Let S be a sieve on X and let R be a τ-covering sieve on X. If for all $f\colon Y \to X$ in R, $f^*(S)$ is τ-covering, then S itself is τ-covering.
(iii) The maximal sieve $よ(X)$ on any X is τ-covering.

A family of maps $(Y_i \to X)_{i\in I}$ is called a *τ-covering family* or a *τ-cover* if it generates a τ-covering sieve. A category equipped with a Grothendieck topology is called a *site*.

Remark 6.14. The following further properties of a Grothendieck topology τ are consequences of the axioms (i)–(iii):

(iv) If R is a τ-covering sieve on X and S is any sieve on X containing R, then S is τ-covering.
(v) Any finite intersection of τ-covering sieves on X is τ-covering.

Definition 6.15 (Sheaf)**.** Let $\mathcal{C}$ be a category and τ a topology on $\mathcal{C}$. A presheaf $F \in \mathrm{P}(\mathcal{C})$ is called a *τ-sheaf* if it satisfies descent along every τ-covering sieve. We denote by $\mathrm{Sh}_\tau(\mathcal{C}) \subset \mathrm{P}(\mathcal{C})$ the full subcategory of τ-sheaves.

Remark 6.16 (Poset of topologies)**.** Let τ and ρ be topologies on $\mathcal{C}$. We say that τ is *coarser* than ρ and that ρ is *finer* than τ, and we write $\tau \leq \rho$, if every τ-covering sieve is a ρ-covering sieve. The collection of topologies on $\mathcal{C}$ is a poset under the relation $\leq$. By definition, any intersection of topologies on $\mathcal{C}$ is again a topology. The poset of topologies on $\mathcal{C}$ therefore admits all infima, and hence also all suprema. In particular, any collection σ of sieves on objects of $\mathcal{C}$ generates a topology $\bar{\sigma}$, which is the coarsest topology on $\mathcal{C}$ containing σ.

Example 6.17. On any category $\mathcal{C}$, we can consider the following topologies:

(i) The *discrete topology* is the finest topology: all sieves are covering. The final presheaf $*$ is the only sheaf in the discrete topology (by Example 6.6).
(ii) The *indiscrete topology* is the coarsest topology: only the maximal sieves are covering. All presheaves are sheaves in the indiscrete topology.

Example 6.18 (Locales and topological spaces)**.** Let O be a locale, viewed as a category. If we define a sieve on $u \in O$ to be covering if its supremum is u, we obtain a topology on O, called the *canonical topology*. Indeed, Conditions (i) and (ii) follow from the distributivity law, and Condition (iii) is clear.

In particular, if T is a topological space, we have the canonical topology on the poset $\mathrm{Open}(T)$. We usually write $\mathrm{Sh}(T)$ for the category of sheaves $\mathrm{Sh}_{\mathrm{can}}(\mathrm{Open}(T))$, which are simply called *sheaves on T*. The covering sieves are exactly the sieves generated by open coverings, so that, by Proposition 6.9, a presheaf $F\colon \mathrm{Open}(T)^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf if and only if, for every $U \in \mathrm{Open}(T)$ and every open covering $(U_i \subset U)_{i\in I}$, the diagram

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)$$

is an equalizer. For example, this holds for the presheaf $\mathrm{Map}_{\mathrm{Top}}(-, X)$ of X-valued continuous functions for any topological space X.

Example 6.19 (Weiss topologies)**.** Let T be a topological space and κ a cardinal. The *κ-Weiss topology* on $\mathrm{Open}(T)$ is defined as follows: a sieve R on U is covering if any subset of U of size $< \kappa$ is contained in some element of R. This recovers the discrete topology for $\kappa = 0$ and the indiscrete topology for $\kappa > |T|$ (Example 6.17). For $\kappa = 1$, the covering sieves are exactly the nonempty sieves, and the corresponding sheaves are the constant presheaves. For $\kappa = 2$, the covering sieves are those of the canonical topology (Example 6.18), except that the empty sieve does not cover the empty set.

Warning 6.20. The notion of Grothendieck topology is *not* a direct generalization of the notion of topology on a set. Example 6.18 explains the relation between the two notions. Historically, Grothendieck named the new concept “topology” because it replaced topological spaces in the context of sheaf theory.

Proposition 6.21 (The adjunction between topologies and subcategories)**.** *Let $\mathcal{C}$ be a category. The map of posets*

$$\{\textit{topologies on } \mathcal{C}\}^{\mathrm{op}} \to \{\textit{subcategories of } \mathrm{P}(\mathcal{C})\},$$
$$\tau \mapsto \mathrm{Sh}_\tau(\mathcal{C}),$$

has a left adjoint $\mathcal{E} \mapsto \tau_\mathcal{E}$, i.e., there exists a finest topology $\tau_\mathcal{E}$ such that $\mathcal{E} \subset \mathrm{Sh}_{\tau_\mathcal{E}}(\mathcal{C})$. A sieve R on X is $\tau_\mathcal{E}$-covering if and only if, for every map $f\colon Y \to X$, every $F \in \mathcal{E}$ satisfies descent along $f^*(R)$.

Corollary 6.22. *Let $(\tau_i)_{i\in I}$ be a family of topologies on $\mathcal{C}$ and let $\tau = \bigvee_{i\in I} \tau_i$ be their supremum. Then*

$$\mathrm{Sh}_\tau(\mathcal{C}) = \bigcap_{i\in I} \mathrm{Sh}_{\tau_i}(\mathcal{C}).$$

Corollary 6.23. *Let $\mathcal{C}$ be a category and let σ be a collection of sieves on $\mathcal{C}$ that is closed under pullbacks (i.e., satisfies Condition* (i) *of Definition 6.13). Let $\bar\sigma$ be the topology generated by σ. Then a presheaf F on $\mathcal{C}$ is a $\bar\sigma$-sheaf if and only if it satisfies descent along all sieves in σ.*

In practice, topologies are often defined by means of covering families rather than covering sieves:

Definition 6.24 (Pretopology)**.** A *pretopology* π on a category $\mathcal{C}$ assigns to each object $X \in \mathcal{C}$ a collection of families of $\mathcal{C}_{/X}$, called *π-covering families* or *π-covers*, subject to the following conditions:

- (i) If $(U_i \to X)_{i\in I}$ is a π-covering family and $f\colon Y \to X$ is any map, then the pullbacks $U_i \times_X Y$ exist in $\mathcal{C}$ and the family $(U_i \times_X Y \to Y)_{i\in I}$ is π-covering.
- (ii) If $(U_i \to X)_{i\in I}$ and $(V_{ij} \to U_i)_{j\in J_i}$ are π-covering families, then $(V_{ij} \to X)_{i\in I, j\in J_i}$ is π-covering.
- (iii) The singleton family $(\mathrm{id}_X\colon X \to X)$ is π-covering.

The *topology associated with* a pretopology π is the coarsest topology for which π-covering families generate covering sieves (which exists by Remark 6.16).

Example 6.25. Let τ be a topology on $\mathcal{C}$. If (and only if) $\mathcal{C}$ has pullbacks, then the τ-covering families form a pretopology on $\mathcal{C}$, whose associated topology is τ. For example, if X is a topological space or an algebraic functor, covering families for the canonical topology on $\mathrm{Open}(X)$ are precisely open coverings, which in particular form a pretopology on $\mathrm{Open}(X)$.

Proposition 6.26 (Pretopologies and descent)**.** *Let $\mathcal{C}$ be a category, let π be a pretopology on $\mathcal{C}$, and let τ be the topology associated with π.*

- (i) *A sieve is τ-covering if and only if it contains a π-covering family.*
- (ii) *A presheaf F is a τ-sheaf if and only if, for every π-covering family $(U_i \to X)_{i\in I}$, the following diagram is an equalizer:*

$$F(X) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_X U_j).$$

Example 6.27 (The standard topology on Top)**.** Open coverings form a pretopology on the category Top of all topological spaces (and many similar categories, like that of smooth manifolds). If τ is the associated topology, Proposition 6.26 implies the following:

- (i) A sieve on X is τ-covering if and only if it contains an open covering of X. Hence, a family $(Y_i \to X)_{i\in I}$ is τ-covering if and only if it locally has sections, i.e., if and only if every point of X has a neighborhood that maps to some Y_i over X.
- (ii) A presheaf $F\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$ is a τ-sheaf if and only if $F|_{\mathrm{Open}(T)}$ is a sheaf on T for each $T \in \mathrm{Top}$.

For example, for every $X \in \mathrm{Top}$, the presheaf $よ(X)$ of X-valued functions is a τ-sheaf (by Example 6.18).

Example 6.28 (The Zariski topology)**.** There is a pretopology on $\mathrm{CAlg}^{\mathrm{op}}$ whose covers are the families $(R \to R_{f_i})_{i\in I}$ such that $(f_i)_{i\in I}$ generates the unit ideal in R. The associated topology is called the *Zariski topology.* By Proposition 6.26, we have the following:

- (i) A sieve $S \subset \mathrm{Spec}(R)$ is Zariski-covering if and only if the set of $f \in R$ such that $R \to R_f$ belongs to S generates the unit ideal. Consequently, an arbitrary family $(R \to R_i)_{i\in I}$ is Zariski-covering if and only if $\coprod_{i\in I} \mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ is a Zariski-local epimorphism (Definition 4.39).
- (ii) An algebraic functor $X\colon \mathrm{CAlg} \to \mathrm{Set}$ is a Zariski sheaf if and only if, for any ring R and any family $(f_i)_{i\in I}$ generating the unit ideal in R, the following diagram is an equalizer:

$$X(R) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_if_j}).$$

For example, any affine scheme is a Zariski sheaf by Corollary 2.72, as is $\mathrm{Proj}(A)$ for any $\mathbb{N}$-graded ring A by Proposition 3.57. Note that if we only consider families $(f_i)_{i\in I}$ where I is finite, we obtain another pretopology on $\mathrm{CAlg}^{\mathrm{op}}$, whose associated topology is *also* the Zariski topology. Hence, the condition in (ii) is *equivalent* to the seemingly weaker condition where we require that I be finite. On the other hand, by Proposition 6.9, it is also equivalent to the following seemingly stronger condition: for every Zariski-covering family $(R \to R_i)_{i\in I}$, the diagram

$$X(R) \to \prod_{i\in I} X(R_i) \rightrightarrows \prod_{i,j\in I} X(R_i \otimes_R R_j)$$

is an equalizer.

Example 6.29 (The fpqc and fppf topologies)**.** The definition of the Zariski topology on $\mathrm{CAlg}^{\mathrm{op}}$ is a special case of a more general construction. Let E be a class of ring maps, which contains isomorphisms and is closed under composition and cobase change. Then there is a pretopology on $\mathrm{CAlg}^{\mathrm{op}}$ whose covers are the families $(R \to R_i)_{i\in I}$ such that:

- I is finite;
- $\coprod_{i\in I} \mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ is surjective;
- each map $R \to R_i$ belongs to E.

For example:

(i) If E is the class of localizations $R \to R_f$, this recovers the Zariski topology of Example 6.28.
(ii) If E is the class of flat maps, the associated topology on $\mathrm{CAlg}^{\mathrm{op}}$ is called the *fpqc topology*.
(iii) If E is the class of flat maps of finite presentation, the associated topology on $\mathrm{CAlg}^{\mathrm{op}}$ is called the *fppf topology*.

Since the localization maps $R \to R_f$ are flat and of finite presentation, we have the following relations between these topologies: $\mathrm{Zar} \leq \mathrm{fppf} \leq \mathrm{fpqc}$.

Remark 6.30 (Finitary topologies)**.** Let τ be a topology on $\mathcal{C}$ associated with a pretopology whose covering families are *finite*. By Proposition 6.26, the condition of being a τ-sheaf can then be expressed using only finite limits. It follows that the subcategory $\mathrm{Sh}_\tau(\mathcal{C}) \subset \mathrm{P}(\mathcal{C})$ is closed under filtered colimits. This applies to the Zariski topology on $\mathrm{CAlg}^{\mathrm{op}}$ and all the topologies from Example 6.29.

Definition 6.31 (Dense subcategory)**.** Let $(\mathcal{C}, \tau)$ be a site. A *dense subcategory* of $(\mathcal{C}, \tau)$ is a full subcategory $\mathcal{D} \subset \mathcal{C}$ such that every $X \in \mathcal{C}$ admits a τ-cover $(U_i \to X)_{i\in I}$ with $U_i \in \mathcal{D}$.

Proposition 6.32 (The "comparison lemma")**.** *Let $(\mathcal{C}, \tau)$ be a site and $\mathcal{D} \subset \mathcal{C}$ a dense subcategory. Let $\tau|_{\mathcal{D}}$ be the collection of sieves $R|_{\mathcal{D}} \subset \text{よ}_{\mathcal{D}}(X)$ for all $X \in \mathcal{D}$ and all τ-covering sieves $R \subset \text{よ}_{\mathcal{C}}(X)$. Then $\tau|_{\mathcal{D}}$ is a topology on $\mathcal{D}$ and the functor $\mathrm{P}(\mathcal{C}) \to \mathrm{P}(\mathcal{D})$, $F \mapsto F|_{\mathcal{D}}$, restricts to an equivalence of categories $\mathrm{Sh}_\tau(\mathcal{C}) \xrightarrow{\sim} \mathrm{Sh}_{\tau|_{\mathcal{D}}}(\mathcal{D})$.*

Example 6.33 (Comparison lemma for presheaves)**.** Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory. Then $\mathcal{D}$ is dense with respect to the indiscrete topology (Example 6.17(ii)) if and only if every object of $\mathcal{C}$ is a retract of an object of $\mathcal{D}$. In this case, the comparison lemma says that $\mathrm{P}(\mathcal{C}) \xrightarrow{\sim} \mathrm{P}(\mathcal{D})$.

Example 6.34 (Basis of a topological space)**.** Let T be a topological space and let $\mathcal{B} \subset \mathrm{Open}(T)$ be a basis of its topology. By definition, $\mathcal{B}$ is dense with respect to the canonical topology on $\mathrm{Open}(T)$, and the restriction of the canonical topology to $\mathcal{B}$ consists of sieves generated by open coverings in $\mathcal{B}$. Hence, the comparison lemma provides an equivalence $\mathrm{Sh}(T) \xrightarrow{\sim} \mathrm{Sh}(\mathcal{B})$.

Example 6.35 (Sheaves on manifolds)**.** Let Man_n be the category of n-dimensional topological manifolds, equipped with the standard topology (induced by the pretopology of open coverings). Since by definition every n-manifold admits an open covering by manifolds homeomorphic to $\mathbb{R}^n$, the full subcategory $\{\mathbb{R}^n\} \subset \mathrm{Man}_n$ spanned by $\mathbb{R}^n$ is dense. By the comparison lemma, $\mathrm{Sh}(\mathrm{Man}_n) \simeq \mathrm{Sh}(\{\mathbb{R}^n\})$.

Example 6.36 (Sheaves on $\mathrm{Spec}(R)$)**.** Let R be a ring. Consider the following two full subcategories of the locale $\mathrm{Open}(\mathrm{Spec}(R)) \simeq \mathrm{Open}(\mathrm{Prim}(R)) \simeq \mathrm{Rad}_R$:

(i) $\mathrm{Open}^{\mathrm{pr}}(\mathrm{Spec}(R))$ consists of open subfunctors of the form $\mathrm{D}(f)$ with $f \in R$ (called the *principal* open subschemes of $\mathrm{Spec}(R)$);
(ii) $\mathrm{Open}^{\mathrm{aff}}(\mathrm{Spec}(R))$ consists of the *affine* open subfunctors of $\mathrm{Spec}(R)$.

We have $\mathrm{Open}^{\mathrm{pr}}(\mathrm{Spec}(R)) \subset \mathrm{Open}^{\mathrm{aff}}(\mathrm{Spec}(R))$, and both subcategories are dense in $\mathrm{Open}(\mathrm{Spec}(R))$, since $\mathrm{D}(I) = \bigvee_{f\in I} \mathrm{D}(f)$. By the comparison lemma, the three categories of sheaves are equivalent. In particular, every sheaf on $\mathrm{Open}^{\mathrm{pr}}(\mathrm{Spec}(R))$ *extends uniquely* to a sheaf on $\mathrm{Open}(\mathrm{Spec}(R))$.

Example 6.37 (Sheaves on presheaves)**.** Let $\mathcal{C}$ be a small category, which we view as a full subcategory of $\mathrm{P}(\mathcal{C})$ via the Yoneda embedding. Any topology τ on $\mathcal{C}$ extends canonically to a topology $\hat{\tau}$ on $\mathrm{P}(\mathcal{C})$ as follows: a sieve R on a presheaf F is $\hat{\tau}$-covering if, for any $よ(X) \to F$, the sieve of all $Y \to X$ such that $よ(Y) \to よ(X) \to F$ is in R is τ-covering. Since every object of $\mathrm{P}(\mathcal{C})$ is a quotient of a coproduct of representable presheaves, the comparison lemma applies to the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{P}(\mathcal{C})$, so that

$$\mathrm{Sh}_\tau(\mathcal{C}) \simeq \mathrm{Sh}_{\hat{\tau}}(\mathrm{P}(\mathcal{C})).$$

(The assumption that $\mathcal{C}$ is small can be removed either by considering $\widehat{\mathrm{Set}}$-valued sheaves or by replacing $\mathrm{P}(\mathcal{C})$ by the full subcategory of presheaves that are *small* colimits of representables, cf. Remark 2.53.)

Remark 6.38. The comparison lemma remains true for sheaves of categories (as in Remark 6.12). Applied to $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}$ and τ the Zariski topology, Example 6.37 shows that a Zariski sheaf of categories on $\mathrm{CAlg}^{\mathrm{op}}$ extends uniquely to a sheaf of categories on $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$. By the proof of the comparison lemma, this extension is given by right Kan extension, i.e., by forming categories of quasi-coherent objects (Definition 4.13). Granted this, Example 6.37 justifies Remark 4.42.

6.3. **Sheafification.**

Theorem 6.39 (Sheafification)**.** *Let $\mathcal{C}$ be a small category and τ a topology on $\mathcal{C}$. Then the inclusion $\mathrm{Sh}_\tau(\mathcal{C}) \subset \mathrm{P}(\mathcal{C})$ admits a left adjoint*

$$\mathrm{a}_\tau \colon \mathrm{P}(\mathcal{C}) \to \mathrm{Sh}_\tau(\mathcal{C}),$$

called τ-sheafification*. Moreover:*

(i) a_τ *is left exact, i.e., preserves finite limits.*
(ii) *A sieve $R \subset よ(X)$ is τ-covering if and only if* $\mathrm{a}_\tau(R) \xrightarrow{\sim} \mathrm{a}_\tau(よ(X))$.

Remark 6.40. The assumption that $\mathcal{C}$ is small can be replaced with the following weaker assumption: for every $X \in \mathcal{C}$, there exists a small collection J of τ-covering sieves on X such that every τ-covering sieve on X contains a sieve in J. This assumption holds for example for the Zariski and fppf topologies on $\mathrm{CAlg}^{\mathrm{op}}$, so that Zariski sheafification and fppf sheafification of algebraic functors make sense. However, it does not hold for the fpqc topology on $\mathrm{CAlg}^{\mathrm{op}}$, and indeed fpqc sheafification does *not* exist.

Corollary 6.41. *Let $(\mathcal{C}, \tau)$ be a small site. Then the category $\mathrm{Sh}_\tau(\mathcal{C})$ admits limits and colimits: limits are computed objectwise, while colimits are computed by sheafifying objectwise colimits.*

Corollary 6.42. *Let $\mathcal{C}$ be a small category. Then there is an isomorphism of posets*

$$\{\textit{topologies on } \mathcal{C}\}^{\mathrm{op}} \xrightarrow{\sim} \{\textit{full subcategories of } \mathrm{P}(\mathcal{C}) \textit{ admitting a left exact left adjoint}\},$$
$$\tau \mapsto \mathrm{Sh}_\tau(\mathcal{C}).$$

Definition 6.43 (Local epimorphism, monomorphism, isomorphism)**.** Let $(\mathcal{C}, \tau)$ be a site, let $f \colon F \to G$ be a map in $\mathrm{P}(\mathcal{C})$, and let $\Delta_f \colon F \to F \times_G F$ be its diagonal.

(i) f is a *τ-local epimorphism* if, for every $X \in \mathcal{C}$ and every $x \in G(X)$, the sieve on X consisting of all $u \colon Y \to X$ such that $u^*(x)$ is in the image of f is a τ-covering sieve.
(ii) f is a *τ-local monomorphism* if its diagonal Δ_f is a τ-local epimorphism.
(iii) f is a *τ-local isomorphism* if it is both a τ-local epimorphism and a τ-local monomorphism.

Proposition 6.44. *Let $(\mathcal{C}, \tau)$ be a small site and let f be a morphism in $\mathrm{P}(\mathcal{C})$. Then $\mathrm{a}_\tau(f)$ is an epimorphism in $\mathrm{Sh}_\tau(\mathcal{C})$ if and only if f is a τ-local epimorphism. The same statement holds for monomorphisms and for isomorphisms.*

Corollary 6.45 (Epi-mono factorization)**.** *Let $(\mathcal{C}, \tau)$ be a small site. Every map $f \colon F \to G$ in $\mathrm{Sh}_\tau(\mathcal{C})$ factors uniquely (up to unique isomorphism) as*

$$F \xrightarrow{f_{\mathrm{epi}}} \mathrm{im}_\tau(f) \xrightarrow{f_{\mathrm{mono}}} G,$$

where f_{epi} is an epimorphism and f_{mono} is a monomorphism. This factorization is obtained by applying a_τ to the factorization $F \to \mathrm{im}(f) \to G$ in $\mathrm{P}(\mathcal{C})$. In particular, f_{epi} is the coequalizer in $\mathrm{Sh}_\tau(\mathcal{C})$ of the two projections $F \times_G F \rightrightarrows F$.

Warning 6.46. A map in $\mathrm{Sh}_\tau(\mathcal{C})$ is a monomorphism in $\mathrm{Sh}_\tau(\mathcal{C})$ if and only if it is a monomorphism in $\mathrm{P}(\mathcal{C})$ (since the inclusion $\mathrm{Sh}_\tau(\mathcal{C}) \hookrightarrow \mathrm{P}(\mathcal{C})$ preserves limits and hence monomorphisms). However, the corresponding statement for epimorphisms does not hold: epimorphisms in $\mathrm{Sh}_\tau(\mathcal{C})$ are usually not epimorphisms in $\mathrm{P}(\mathcal{C})$, i.e., they are not objectwise surjective. For example, if R is a ring and $(f, g) = R$, then $\mathrm{Spec}(R_f \times R_g) \to \mathrm{Spec}(R)$ is an epimorphism in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$, since it is a Zariski-local epimorphism, but it is not objectwise surjective.

In the case of sheaves on a topological space, we can rephrase the criterion of Proposition 6.44 in terms of *stalks*:

Definition 6.47 (Stalk)**.** Let T be a topological space, let $x \in T$, and let $F \in \mathrm{P}(\mathrm{Open}(T))$. The *stalk* of F at x is the set

$$F_x = \operatorname*{colim}_{x \in U} F(U),$$

where the colimit is indexed by the poset of open neighborhoods of x.

Remark 6.48. Since the colimit in the definition of the stalk F_x is a filtered colimit, the functor

$$\mathrm{P}(\mathrm{Open}(T)) \to \mathrm{Set}, \quad F \mapsto F_x,$$

preserves colimits and finite limits.

Proposition 6.49 (Stalkwise characterization of epimorphisms, monomorphisms, and isomorphisms)**.** *Let T be a topological space, let $f\colon F \to G$ be a map in $\mathrm{P}(\mathrm{Open}(T))$, and let $\mathrm{a}\colon \mathrm{P}(\mathrm{Open}(T)) \to \mathrm{Sh}(T)$ be the sheafification functor. Then $\mathrm{a}(f)$ is an epimorphism (resp. a monomorphism, an isomorphism) in $\mathrm{Sh}(T)$ if and only if, for every $x \in T$, the induced map of stalks $f_x\colon F_x \to G_x$ is surjective (resp. injective, bijective).*

Corollary 6.50. *Let T be a topological space and let $F \in \mathrm{P}(\mathrm{Open}(T))$. Then the unit map $F \to \mathrm{a}(F)$ induces a bijection on all stalks.*

Remark 6.51 (Underlying space and Zariski sheafification)**.** Let X be an algebraic functor. By Corollary 5.35 and Proposition 6.44, the unit map $X \to \mathrm{a}_{\mathrm{Zar}}(X)$ induces an isomorphism of topological spaces $|X| \xrightarrow{\sim} |\mathrm{a}_{\mathrm{Zar}}(X)|$. This implies that there is a factorization

$$\begin{array}{ccc} \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) & \xrightarrow{|-|} & \mathrm{Top}, \\ {\scriptstyle \mathrm{a}_{\mathrm{Zar}}}\downarrow & \nearrow & \\ \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}) & & \end{array}$$

where the dashed arrow is colimit-preserving.

7. Schemes

7.1. **The category of schemes.**

Definition 7.1 (Scheme)**.** A *scheme* is a functor $X\colon \mathrm{CAlg} \to \mathrm{Set}$ satisfying the following two conditions:

(i) X is a sheaf for the Zariski topology on $\mathrm{CAlg}^{\mathrm{op}}$ (Example 6.28);
(ii) X admits an open covering by affine schemes (Definition 5.40).

We denote by $\mathrm{Sch} \subset \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ the full subcategory spanned by schemes. For any scheme S, we denote by Sch_S the slice category $\mathrm{Sch}_{/S}$, whose objects are called *S-schemes*. When $S = \mathrm{Spec}(k)$ for some ring k, an S-scheme is also called a *k-scheme*.

Our goal in this section is to investigate the closure properties of the category of schemes inside the larger category of Zariski sheaves.

Proposition 7.2. *If X is a scheme and $Y \subset X$ is a locally closed subfunctor, then Y is a scheme.*

Example 7.3 (Examples of schemes)**.**

(i) For any ring A, the affine scheme $\mathrm{Spec}(A)$ is a scheme, since it is a Zariski sheaf by Corollary 2.72. By Proposition 7.2, every quasi-affine scheme is also a scheme.
(ii) For any $\mathbb{N}$-graded ring A, $\mathrm{Proj}(A)$ is a scheme: it is a Zariski sheaf by Proposition 3.57, and it admits an open covering by the affine schemes $\mathrm{D}(f) \simeq \mathrm{Spec}(A_{(f)})$ for $f \in A_+$ homogeneous (Example 5.42(iii)). In particular, for any ring k, every projective k-scheme is a scheme. By Proposition 7.2, every quasi-projective k-scheme is also a scheme.
(iii) The affine line with doubled origin (Example 4.47) is a scheme: it admits an open covering by two copies of $\mathbb{A}^1$, and it follows from Zariski descent for modules that it is a Zariski sheaf.
(iv) We have the following relative generalization of (i) and (ii): if X is a scheme and $Y \to X$ is either a quasi-affine morphism (Definition 4.34) or a quasi-projective morphism (Definition 4.35), then Y is also a scheme.

(v) Let k be a ring, M a k-module, and $n \in \mathbb{N}$. Proposition 7.2 and the Plücker embedding from Example 3.100 show that the Grassmannian $\mathrm{Gr}_n(M)$ is a scheme. More generally, for any increasing sequence of natural numbers $n = (n_1, \dots, n_s)$, the flag scheme $\mathrm{Flag}_n(M)$ is a scheme, being a closed subfunctor of the projective space $\mathbb{P}\big(\bigotimes_{i=1}^s \Lambda_k^{n_i} M\big)$ by Examples 3.103, 3.100, and 3.95.

Remark 7.4 (Residue fields of schemes)**.** Let X be a scheme and let $x \in |X|$. By Example 5.38(i) and Proposition 5.37, the residue field $\kappa(x)$ of X at x exists, and it coincides with the residue field at x of any open subscheme $U \subset X$ containing x. Moreover, by taking U to be affine, we see that the canonical map $\mathrm{Spec}(\kappa(x)) \to X$ is a monomorphism.

Definition 7.5 (Equivalence relation)**.** Let $\mathcal{C}$ be a category. An *equivalence relation* in $\mathcal{C}$ is a diagram of the form

$$R \rightrightarrows X$$

such that, for every object $Y \in \mathcal{C}$, the induced map $\mathrm{Map}(Y, R) \to \mathrm{Map}(Y, X) \times \mathrm{Map}(Y, X)$ is injective and defines an equivalence relation on $\mathrm{Map}(Y, X)$. The coequalizer of the diagram, if it exists, is called the *quotient* of X by R and denoted by X/R.

Remark 7.6 (Effective epimorphism)**.** In a category with pullbacks, any map $f\colon X \to Y$ defines an equivalence relation $X \times_Y X \rightrightarrows X$. The map f is called an *effective epimorphism* if it is recovered by taking the quotient of this relation. In the category Set, and hence in any category of sheaves $\mathrm{Sh}_\tau(\mathcal{C})$, all epimorphisms are effective (Corollary 6.45), but this is not true in general. In Top, epimorphisms are surjective maps, while effective epimorphisms are submersive maps. In CAlg, effective epimorphisms are surjective maps, but there are non-surjective epimorphisms, such as $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

Definition 7.7 (Locally trivial relation)**.** An equivalence relation $s, t\colon R \rightrightarrows X$ in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ is called *locally trivial* if there is an open covering $(U_i \subset X)_{i \in I}$ such that the maps

$$s^{-1}(U_i) \hookrightarrow R \xrightarrow{t} X$$

are open immersions.

Lemma 7.8. *Let $(X_i)_{i\in I}$ be a family in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ with coproduct X. Then the canonical maps $X_i \hookrightarrow X$ are clopen immersions. In particular, $(X_i \hookrightarrow X)_{i\in I}$ is an open covering of X.*

Example 7.9 (Gluing datum)**.** A *gluing datum* in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ consists of families of objects $(U_i)_{i\in I}$ and $(U_{ij})_{i,j\in I}$ and maps $U_{ij} \to U_i \times U_j$ such that:

(i) for all $i, j \in I$, the maps $U_{ij} \to U_i$ and $U_{ij} \to U_j$ are open immersions;
(ii) the induced diagram
$$\coprod_{i,j\in I} U_{ij} \rightrightarrows \coprod_{i\in I} U_i$$
in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ is an equivalence relation.

By Lemma 7.8, the relation in (ii) is then locally trivial. Condition (ii) means explicitly the following:

- (Reflexivity) The diagonal map $U_i \to U_i \times U_i$ lands in U_{ii}.
- (Symmetry) The map $U_{ij} \to U_i \times U_j \xrightarrow{\sim} U_j \times U_i$ lands in U_{ji}.
- (Transitivity) The map $U_{ij} \times_{U_j} U_{jk} \to U_i \times U_k$ lands in U_{ik}.

If X is the quotient of the relation in (ii), then the maps $U_i \to X$ are open immersions and form an open covering of X such that $U_{ij} \xrightarrow{\sim} U_i \times_X U_j$. Conversely, given a Zariski sheaf X and an open covering $(U_i \subset X)_{i\in I}$, we obtain a gluing datum by setting $U_{ij} = U_i \cap U_j$, such that the quotient of the associated relation is X.

Theorem 7.10 (Limits and colimits of schemes)**.** *The full subcategory* $\mathrm{Sch} \subset \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ *is closed under the following constructions:*

(i) *finite limits;*
(ii) *limits of cofiltered diagrams with affine transition maps;*
(iii) *arbitrary coproducts;*
(iv) *quotients of locally trivial equivalence relations.*

Example 7.11 (The initial scheme)**.** Since the empty sieve on the zero ring is Zariski-covering, every Zariski sheaf X satisfies $X(0) = *$ (Example 6.6). Hence, the empty scheme $\varnothing = \mathrm{Spec}(0)$ (Example 2.30) is the initial object in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ and in Sch.

Example 7.12 (Coproducts of affine schemes)**.** Let R and S be rings. In CAlg, the following square is a pushout square:

$$\begin{array}{ccc} R\times S & \longrightarrow & S \\ \downarrow & & \downarrow \\ R & \longrightarrow & 0. \end{array}$$

By Example 6.7, every Zariski sheaf X satisfies $X(R\times S) \xrightarrow{\sim} X(R)\times X(S)$. This means that

$$\mathrm{Spec}(R)\sqcup \mathrm{Spec}(S) \xrightarrow{\sim} \mathrm{Spec}(R\times S)$$

in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ and hence in Sch.

Warning 7.13. By Examples 7.11 and 7.12, the inclusion Aff $\subset$ Sch preserves finite coproducts. Note however that it does not preserve infinite coproducts: given an infinite family of nonzero rings $(R_i)_{i\in I}$, the scheme $X = \coprod_{i\in I}\mathrm{Spec}(R_i)$ is not affine. Indeed, X admits an open covering by the subschemes $\mathrm{D}(e_i) = \mathrm{Spec}(R_i)$, but the family of functions $(e_i)_{i\in I}$ does not generate the unit ideal in $\mathcal{O}(X) = \prod_{i\in I} R_i$.

Example 7.14 (Coproducts of projective spaces)**.** Let k be a ring and let M and N be k-modules. The two projections $M\oplus N \twoheadrightarrow M$ and $M\oplus N \twoheadrightarrow N$ induce a canonical map of k-schemes

$$\vartheta\colon \mathbb{P}(M)\sqcup\mathbb{P}(N)\to\mathbb{P}(M\oplus N),$$

which is an "additive" analogue of the Segre embedding (Example 3.95). If $M = k^{(I)}$ and $N = k^{(J)}$, this is a map

$$\vartheta\colon \mathbb{P}^I_k\sqcup\mathbb{P}^J_k\to\mathbb{P}^{I\sqcup J}_k.$$

If $M = k^{m+1}$ and $N = k^{n+1}$, this becomes the map

$$\vartheta\colon \mathbb{P}^m_k\sqcup\mathbb{P}^n_k\to\mathbb{P}^{m+n+1}_k,$$

which is given in coordinates by the formulas

$$[x_0:\ldots:x_m]\mapsto[x_0:\ldots:x_m:0:\ldots:0],$$
$$[y_0:\ldots:y_n]\mapsto[0:\ldots:0:y_0:\ldots:y_n].$$

We claim that ϑ is a closed immersion. If $(e_i)_{i\in I}$ and $(f_j)_{j\in J}$ are generating families of M and N, then $\mathbb{P}(M\oplus N)$ has an open covering by the subschemes $\mathrm{D}(e_i,0)$ and $\mathrm{D}(0,f_j)$. As ϑ is a map of Zariski sheaves, it suffices by Proposition 2.79 to show that the pullback of ϑ to these open subschemes is a closed immersion. Since the preimage of $\mathrm{D}(e_i,0)$ in $\mathbb{P}(N)$ is the empty scheme $\mathrm{D}(0)$, the pullback of ϑ to $\mathrm{D}(e_i,0)$ is the same as the pullback of $\vartheta|_{\mathbb{P}(M)}$, which is a closed immersion. Similarly, the pullback of ϑ to $\mathrm{D}(0,f_j)$ is also the pullback of the closed immersion $\vartheta|_{\mathbb{P}(N)}$, which proves the claim. More generally, there is a canononical monomorphism of k-schemes $\coprod_{i\in I}\mathbb{P}(M_i)\hookrightarrow\mathbb{P}\big(\bigoplus_{i\in I}M_i\big)$ for any family of k-modules $(M_i)_{i\in I}$, which is a closed immersion when I is finite.

Proposition 7.15. *Let S be a scheme. Then the subcategories*

$$\mathrm{Aff}_S\subset\mathrm{QAff}_S\subset\mathrm{Sch}_S\supset\mathrm{QProj}_S\supset\mathrm{Proj}_S$$

of affine, quasi-affine, projective, and quasi-projective S-schemes are closed under finite limits and finite coproducts.

Proposition 7.16 (Schemes as colimits of affine schemes)**.** *Let X be a scheme and let $(U_i\subset X)_{i\in I}$ and $(U_{ijk}\subset U_i\cap U_j)_{k\in K(i,j)}$ be open coverings by affine schemes. Then the diagram*

$$\coprod_{i,j,k}U_{ijk}\rightrightarrows\coprod_i U_i\to X$$

is a coequalizer in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ *and hence in* Sch.

Example 7.17 (Proj via gluing)**.** Let A be an $\mathbb{N}$-graded ring and let $I\subset A_+$ be a homogeneous subset such that $A_+\subset\sqrt{(I)}$, so that $(\mathrm{D}(f)\subset\mathrm{Proj}(A))_{f\in I}$ is an open covering. Since $\mathrm{D}(f)\simeq\mathrm{Spec}(A_{(f)})$ and $\mathrm{D}(f)\cap\mathrm{D}(g)=\mathrm{D}(fg)$, we obtain a coequalizer diagram

$$\coprod_{f,g\in I}\mathrm{Spec}(A_{(fg)})\rightrightarrows\coprod_{f\in I}\mathrm{Spec}(A_{(f)})\to\mathrm{Proj}(A)$$

in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$, which presents $\mathrm{Proj}(A)$ as a gluing of the affine schemes $\mathrm{Spec}(A_{(f)})$.

Example 7.18 (Projective space via gluing)**.** Let $n \geq -1$ and let us specialize Example 7.17 to the case $A = \mathbb{Z}[x_0, \dots, x_n]$, so that $\mathrm{Proj}(A) = \mathbb{P}^n$. Recall that $\mathbb{P}^n(R)$ is the set of quotient lines of $R^{\{0,\dots,n\}}$. For $i \in \{0, \dots, n\}$, $\mathrm{D}(x_i) \subset \mathbb{P}^n$ is the subfunctor consisting of quotient lines $a\colon R^{\{0,\dots,n\}} \twoheadrightarrow L$ such that a_i is an isomorphism, which is isomorphic to $\mathbb{A}^n$. The intersection $\mathrm{D}(x_i) \cap \mathrm{D}(x_j) = \mathrm{D}(x_i x_j)$ is the subfunctor where both a_i and a_j are isomorphisms, which is isomorphic to $\mathbb{A}^{n-1} \times \mathbb{G}_m$ if $i \neq j$. We therefore have a coequalizer in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$ of the form

$$\coprod_{i<j} (\mathbb{A}^{n-1} \times \mathbb{G}_m) \rightrightarrows \coprod_{i=0}^{n} \mathbb{A}^n \to \mathbb{P}^n.$$

Example 7.19. Let X be the affine line with doubled origin (see Example 4.47). Similarly to $\mathbb{P}^1$, X is obtained by gluing two copies of $\mathbb{A}^1$ along a copy of $\mathbb{G}_\mathrm{m}$. However, the gluing data for X and for $\mathbb{P}^1$ are different: if $u\colon \mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{A}^1$ is the open immersion given by $t \mapsto t$ and $v\colon \mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{A}^1$ is the open immersion given by $t \mapsto t^{-1}$, then we have the following pushout squares of open immersions in Sch:

$$\begin{array}{ccc} \mathbb{G}_\mathrm{m} & \xrightarrow{u} & \mathbb{A}^1 \\ {\scriptstyle v}\downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1, \end{array} \qquad \begin{array}{ccc} \mathbb{G}_\mathrm{m} & \xrightarrow{u} & \mathbb{A}^1 \\ {\scriptstyle u}\downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & X. \end{array}$$

Example 7.20 (Projective space via group actions)**.** Let $n \geq 0$. The affine group scheme $\mathbb{G}_\mathrm{m}$ acts on the punctured affine space $\mathbb{A}^n - 0$ by scaling. By Proposition 3.24, the canonical map $(\mathbb{A}^n - 0)/\mathbb{G}_\mathrm{m} \to \mathbb{P}^{n-1}$ is a monomorphism as well as a Zariski-local epimorphism. By Proposition 6.44, it is thus an *isomorphism* if we compute the quotient in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$. In other words, the diagram

$$\mathbb{G}_\mathrm{m} \times (\mathbb{A}^n - 0) \underset{\text{project}}{\overset{\text{act}}{\rightrightarrows}} \mathbb{A}^n - 0 \longrightarrow \mathbb{P}^{n-1}$$

is a coequalizer in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$. Since it is a diagram of schemes, it is also a coequalizer in the category of schemes, i.e., $\mathbb{P}^{n-1}$ is the quotient of $\mathbb{A}^n - 0$ by the scaling action of $\mathbb{G}_\mathrm{m}$ in the category of schemes. More generally, if A is an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$, then $\mathrm{Proj}(A) \simeq \mathrm{D}(A_+)/\mathbb{G}_\mathrm{m}$ as Zariski sheaves and hence as schemes, where $\mathrm{D}(A_+) \subset \mathrm{Spec}(A)$ (see Remark 3.63). For A an arbitrary $\mathbb{N}$-graded ring, one can show that $\mathrm{Proj}(A)$ is still the quotient $\mathrm{D}(A_+)/\mathbb{G}_\mathrm{m}$ in the category of schemes, even though this is not true anymore in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$.

Remark 7.21 (Zariski topology on the category of schemes)**.** By Proposition 7.2, any open subfunctor of a scheme is a scheme. Hence, open coverings form a pretopology on Sch, whose associated topology is called the *Zariski topology.* The inclusion $\mathrm{CAlg}^{\mathrm{op}} \simeq \mathrm{Aff} \subset \mathrm{Sch}$ satisfies the assumption of the comparison lemma (Proposition 6.32), so that the restriction functor $\mathrm{P}(\mathrm{Sch}) \to \mathrm{P}(\mathrm{CAlg}^{\mathrm{op}})$ induces an equivalence of categories

$$\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sch}) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}).$$

7.2. **Separatedness.**

Definition 7.22 (Locally separated, separated)**.** A morphism of algebraic functors $f\colon Y \to X$ is *locally separated* if its diagonal $\Delta_f\colon Y \to Y \times_X Y$ is an immersion, and it is *separated* if Δ_f is a closed immersion. An algebraic functor X is *locally separated* (resp. *separated*) if the unique map $X \to \mathrm{Spec}(\mathbb{Z})$ is, i.e., if the diagonal $\Delta_X\colon X \to X \times X$ is an immersion (resp. a closed immersion).

For schemes, "locally separated" turns out to be vacuous, so this notion is rarely used:

Proposition 7.23 (Diagonal of schemes)**.** *Any morphism of schemes $f\colon Y \to X$ is locally separated, i.e., its diagonal $\Delta_f\colon Y \to Y \times_X Y$ is an immersion.*

Proposition 7.24 (Closure properties of separated morphisms)**.**

(i) *Consider a commutative triangle of algebraic functors*

$$\begin{array}{ccccc} Z & & \xrightarrow{g} & & Y \\ & {\scriptstyle h}\searrow & & \swarrow{\scriptstyle f} & \\ & & X. & & \end{array}$$

If f and g are separated, so is h. If h is separated, then so is g. If h is a closed immersion and f is separated, then g is a closed immersion. The same results hold with "locally separated" and "immersion".

(ii) *Consider a cartesian square of algebraic functors*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X. \end{array}$$

If f is (locally) separated, so is f'. The converse holds if X and Y are Zariski sheaves and g is a Zariski-local epimorphism.

Corollary 7.25 (Sections of morphisms of schemes)**.** *Let $f\colon X \to S$ be a morphism of schemes and $s\colon S \to X$ a section of f. Then s is an immersion. If f is separated, then s is a closed immersion.*

Example 7.26.

(i) Any monomorphism is separated, since its diagonal is an isomorphism.
(ii) Any affine scheme is separated: the diagonal of $\mathrm{Spec}(R)$ can be identified with Spec of the multiplication map $R \otimes R \to R$ (which is the diagonal of R in $\mathrm{CAlg}^{\mathrm{op}}$), which is surjective. More generally, any affine morphism and hence by (i) any quasi-affine morphism is separated.
(iii) If A is an $\mathbb{N}$-graded ring, then $\mathrm{Proj}(A)$ is separated: using the standard affine open covering of $\mathrm{Proj}(A)$ and Proposition 7.24(ii), this follows from the observation that the multiplication maps $A_{(f)} \otimes A_{(g)} \to A_{(fg)}$ are surjective. More generally, if A is a quasi-coherent $\mathbb{N}$-graded algebra over X, then $\mathrm{Proj}(A) \to X$ is separated. In particular, any projective morphism and hence by (i) any quasi-projective morphism is separated.
(iv) The affine line with doubled origin (Example 4.47) is not separated.

7.3. **Quasi-compactness and quasi-separatedness.** By Proposition 7.16, any scheme can be written as the colimit of a diagram of affine schemes and open immersions. A scheme is called *quasi-compact and quasi-separated*, or *qcqs* for short, if a *finite* such diagram exists. In this section, we discuss in more detail this finiteness condition, which makes sense more generally for algebraic functors.

Definition 7.27 (Quasi-compact and quasi-separated)**.**

(i) An algebraic functor X is called *quasi-compact* if, for every Zariski-local epimorphism $\coprod_{i\in I} Y_i \to X$, there exists a finite subset $J \subset I$ such that $\coprod_{i\in J} Y_i \to X$ is a Zariski-local epimorphism.
(ii) A morphism of algebraic functors $f\colon Y \to X$ is called *quasi-compact* if, for every morphism $Z \to X$ with Z quasi-compact, the pullback $Y \times_X Z$ is quasi-compact.
(iii) An algebraic functor X is called *quasi-separated* if, for every diagram $Y \to X \leftarrow Z$ with Y and Z quasi-compact, the pullback $Y \times_X Z$ is quasi-compact.
(iv) A morphism of algebraic functors $f\colon Y \to X$ is called *quasi-separated* if its diagonal $\Delta_f\colon Y \to Y \times_X Y$ is quasi-compact.

Proposition 7.28 (Characterizations of quasi-compactness)**.** *For an algebraic functor X, the following conditions are equivalent:*

(i) *X is quasi-compact;*
(ii) *the unique map $X \to \mathrm{Spec}(\mathbb{Z})$ is quasi-compact;*
(iii) *there exists a Zariski-local epimorphism $\coprod_{i\in I} Y_i \to X$ where I is finite and each Y_i affine.*

Proposition 7.29 (Characterizations of quasi-separatedness)**.** *For an algebraic functor X, the following conditions are equivalent:*

(i) *X is quasi-separated;*
(ii) *the unique map $X \to \mathrm{Spec}(\mathbb{Z})$ is quasi-separated;*
(iii) *for all maps $Y \to X$ and $Z \to X$ where Y and Z are affine, $Y \times_X Z$ is quasi-compact.*

Proposition 7.30 (Composition of qc/qs morphisms)**.** *Consider a commutative triangle of algebraic functors*

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & X. & \end{array}$$

(i) *If f and g are quasi-compact, then h is quasi-compact.*
(ii) *If f and g are quasi-separated, then h is quasi-separated.*
(iii) *If h is quasi-compact and f is quasi-separated, then g is quasi-compact.*
(iv) *If h is quasi-separated, then g is quasi-separated.*

(v) *If h is quasi-compact and g is a Zariski-local epimorphism, then f is quasi-compact.*
(vi) *If h is quasi-separated and g is quasi-compact and a Zariski-local epimorphism, then f is quasi-separated.*

Proposition 7.31 (Base change of qc/qs morphisms)**.** *Consider a cartesian square of algebraic functors*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X. \end{array}$$

If f is quasi-compact, then f' is quasi-compact. The converse holds if g is a Zariski-local epimorphism. The same statements hold with "quasi-separated".

Example 7.32 (Quasi-compact immersions)**.**

(i) By Proposition 7.28(iii), a closed subfunctor of a quasi-compact algebraic functor is again quasi-compact. It follows that any closed immersion is quasi-compact. In particular, as the terminology suggests, any separated morphism is quasi-separated.
(ii) Let R be a ring and let $I \subset R$. Considering the open covering $(\mathrm{D}(f) \subset \mathrm{D}(I))_{f\in I}$ of $\mathrm{D}(I)$, we see that the scheme $\mathrm{D}(I)$ is quasi-compact if and only if the radical ideal $\sqrt{(I)}$ is finitely generated (i.e., equals $\sqrt{(J)}$ for some finite set J). Hence, an open immersion is quasi-compact if and only if the corresponding quasi-coherent radical ideal is finitely generated.
(iii) Any monomorphism is quasi-separated, since its diagonal is an isomorphism.

Example 7.33 (Quasi-affine and quasi-projective schemes)**.**

(i) By Example 7.32(ii), any quasi-affine scheme is quasi-compact. It is also quasi-separated, since it is separated by Example 7.26(ii).
(ii) Let A be an $\mathbb{N}$-graded ring and let $I \subset A$ be a homogeneous subset. As in Example 7.32(ii), $\mathrm{D}(I)$ is quasi-compact if and only if the saturated radical ideal generated by I is finitely generated. In particular, for any ring k, any quasi-projective k-scheme is quasi-compact. It is also quasi-separated, since it is separated by Example 7.26(iii).
(iii) It follows from (i,ii) and Proposition 7.31 that any quasi-affine or quasi-projective morphism of algebraic functors is quasi-compact and quasi-separated.

For schemes, these finiteness conditions depend only on the underlying topological space, and they can be checked using any affine open covering:

Proposition 7.34 (Quasi-compactness for schemes)**.** *Let X be a scheme and $(U_i \subset X)_{i\in I}$ an open covering by affine schemes. The following are equivalent:*

(i) *X is quasi-compact;*
(ii) *the topological space $|X|$ is quasi-compact;*
(iii) *there exists a finite subset $J \subset I$ such that $(U_i \subset X)_{i\in J}$ is an open covering of X.*

Proposition 7.35 (Quasi-separatedness for schemes)**.** *Let X be a scheme and $(U_i \subset X)_{i\in I}$ an open covering by affine schemes. The following are equivalent:*

(i) *X is quasi-separated;*
(ii) *the intersection of any two quasi-compact open subsets of $|X|$ is quasi-compact;*
(iii) *for every $i, j \in I$, the intersection $U_i \cap U_j$ is quasi-compact.*

Remark 7.36. By Propositions 7.34 and 7.35, a scheme is quasi-compact and quasi-separated (often abbreviated as *qcqs*) if and only if it can be written as a *finite* colimit of affine schemes as in Proposition 7.16. This finiteness property propagates to many invariants: for example, the underlying space of a qcqs scheme is spectral, and its category of quasi-coherent modules is compactly generated.

As Example 7.33 shows, most schemes of interest are qcqs. Examples of schemes that are not quasi-compact are infinite coproducts of nonempty schemes and projective spaces $\mathbb{P}^I_k$ where I is an infinite set and k a nonzero ring. An infinite-dimensional affine space with doubled origin is an example of a quasi-compact scheme that is not quasi-separated.

The following standard definition combines the local "algebraic" finiteness properties of §2.9 with the global "topological" finiteness properties of this section:

Definition 7.37 (Morphism of finite presentation/type)**.** Let $f: X \to S$ be a map of algebraic functors.

(i) f is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.
(ii) f is *of finite type* if it is locally of finite type and quasi-compact.

For example, any quasi-projective morphism is of finite type.

7.4. **Locally ringed spaces.** If $f: X \to Y$ is a continuous map of topological spaces, the induced map of posets $f^{-1}: \mathrm{Open}(Y) \to \mathrm{Open}(X)$ preserves open coverings. It follows that the pushforward functor

$$f_* = (f^{-1})^*: \mathrm{P}(\mathrm{Open}(X)) \to \mathrm{P}(\mathrm{Open}(Y)), \quad f_*(F)(V) = F(f^{-1}(U)),$$

preserves sheaves:

$$\begin{array}{ccc} \mathrm{P}(\mathrm{Open}(X)) & \xrightarrow{f_*} & \mathrm{P}(\mathrm{Open}(Y)) \\ \uparrow & & \uparrow \\ \mathrm{Sh}(X) & \overset{f_*}{\dashrightarrow} & \mathrm{Sh}(Y). \end{array}$$

Proposition 7.38 (Pullbacks of sheaves)**.** *Let $f: X \to Y$ be a continuous map. Then the pushforward functor $f_*: \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ admits a left exact left adjoint $f^*: \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. Moreover, for any $F \in \mathrm{Sh}(Y)$ and any $x \in X$, there is a canonical isomorphism of stalks $f^*(F)_x \simeq F_{f(x)}$.*

Remark 7.39. Let $f: X \to Y$ be a continuous map. Since both functors f_* and f^* preserve finite products, they preserve algebraic structures (see Remark 2.13). Hence, they induce adjunctions between categories of sheaves of monoids, groups, rings, etc.

Definition 7.40 (Ringed space)**.** Let k be a ring. A *k-ringed space* $(X, \mathcal{O}_X)$ is a topological space X together with a sheaf of k-algebras $\mathcal{O}_X$ on X. A morphism of k-ringed spaces $(f, \varphi): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f: X \to Y$ together with a map of sheaves of k-algebras $\varphi: \mathcal{O}_Y \to f_*(\mathcal{O}_X)$.

We denote by RS_k the category of k-ringed spaces. When $k = \mathbb{Z}$, we write $\mathrm{RS} = \mathrm{RS}_{\mathbb{Z}}$ for the category of *ringed spaces.*

Remark 7.41. By Proposition 7.38, a map of sheaves of k-algebras $\varphi: \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is equivalently a map of sheaves of k-algebras $f^*(\mathcal{O}_Y) \to \mathcal{O}_X$. Moreover, for any point $x \in X$, the induced map on stalks has the form

$$\varphi_x: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}.$$

Definition 7.42 (Local map)**.** A ring map $\varphi: A \to B$ is *local* if it detects units, i.e., if $A^\times = \varphi^{-1}(B^\times)$.

Remark 7.43 (Local maps between local rings)**.** Let A and B be local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$. For a ring map $\varphi: A \to B$, the following are equivalent:

(i) φ is local;
(ii) $\mathfrak{m}_A = \varphi^{-1}(\mathfrak{m}_B)$;
(iii) $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$.

Definition 7.44 (Locally ringed space)**.** Let k be a ring. A *locally k-ringed space* is a k-ringed space $(X, \mathcal{O}_X)$ such that, for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally k-ringed spaces, a morphism of locally k-ringed spaces $(f, \varphi): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of k-ringed spaces such that, for each $x \in X$, the map $\varphi_x: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We denote by LRS_k the category of locally k-ringed spaces, which is a (non-full!) subcategory of RS_k. When $k = \mathbb{Z}$, we write $\mathrm{LRS} = \mathrm{LRS}_{\mathbb{Z}}$ for the category of *locally ringed spaces.*

Remark 7.45 (Alternative description of locally ringed spaces)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. Given an open $U \subset X$ and a section $s \in \mathcal{O}_X(U)$, let

$$U_s = \{x \in U \mid s_x \in \mathcal{O}_{X,x} \text{ is a unit}\}.$$

By definition of the stalk, U_s is an open subset of U. Moreover, since $\mathcal{O}_X$ is a sheaf, the restriction of s to U_s is a unit, so that U_s is the largest open subset of U with this property. Recall that a ring R is local if and only if the non-units form a subgroup of $(R, +)$, and a map of rings is local if and only if it sends non-units to non-units. We deduce the following description of the subcategory $\mathrm{LRS} \subset \mathrm{RS}$:

(i) A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if and only if $X_0 = X$ and for all open subsets $U \subset X$ and sections $s, t \in \mathcal{O}_X(U)$, $U_{s+t} \subset U_s \cup U_t$.

(ii) If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces and $(f, \varphi)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map in RS, then (f, φ) is a map in LRS if and only if, for every open $V \subset Y$ and every section $t \in \mathcal{O}_Y(V)$, we have $f^{-1}(V_s) = f^{-1}(V)_{\varphi(s)}$.

This alternative description makes no reference to the points of the underlying spaces and can be used to define the category of *locally ringed locales*, which enlarges the category of locally ringed sober spaces.

It turns out that the concept of locally ringed space unifies many kinds of geometric objects in mathematics. To exemplify this, we introduce some notation:

Notation 7.46. Let k be a ring and let $\mathcal{M}$ be a collection of locally k-ringed spaces. We denote by $\mathrm{LRS}_k(\mathcal{M}) \subset \mathrm{LRS}_k$ the full subcategory of locally k-ringed spaces $(X, \mathcal{O}_X)$ with the following property: every $x \in X$ admits an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an object of $\mathcal{M}$.

Example 7.47 (Manifolds as locally ringed spaces)**.**

(i) Let $r \in \mathbb{N} \cup \{\infty, \omega\}$. For an open subset $U \subset \mathbb{R}^n$, denote by $\mathcal{C}^r_U$ the sheaf of ($\mathbb{R}$-valued) functions of class C^r on U. Let $\mathcal{D}\mathrm{isk}_{\mathrm{C}^r}$ be the collection of locally $\mathbb{R}$-ringed spaces $(U, \mathcal{C}^r_U)$ where U is the open unit disk in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Then there are equivalences of categories

$$\mathrm{LRS}(\mathcal{D}\mathrm{isk}_{\mathrm{C}^r}) \simeq \mathrm{LRS}_{\mathbb{R}}(\mathcal{D}\mathrm{isk}_{\mathrm{C}^r}) \simeq \{\mathrm{C}^r\text{-manifolds}\}.$$

In particular, we get *topological manifolds* for $r = 0$, *smooth manifolds* for $r = \infty$, and *analytic manifolds* for $r = \omega$.[7]

(ii) For an open subset $U \subset \mathbb{C}^n$, denote by $\mathcal{H}_U$ the sheaf of ($\mathbb{C}$-valued) holomorphic functions on U. Let $\mathcal{D}\mathrm{isk}_{\mathbb{C}}$ be the collection of locally $\mathbb{C}$-ringed spaces $(U, \mathcal{H}_U)$ where U is the open unit disk in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. Then there is an equivalence of categories

$$\mathrm{LRS}_{\mathbb{C}}(\mathcal{D}\mathrm{isk}_{\mathbb{C}}) \simeq \{\text{complex manifolds}\}.$$

(iii) More generally, many categories of manifolds with boundary, with corners, and with various types of singularities are equivalent to categories of the form $\mathrm{LRS}_{\mathbb{R}/\mathbb{C}}(\mathcal{M})$.

Remark 7.48 (Colimits and limits of ringed spaces)**.** The category RS_k admits colimits and limits. If $(X_i, \mathcal{O}_{X_i})_{i\in\mathcal{I}}$ is a diagram of k-ringed spaces, and if $\iota_i\colon X_i \to \operatorname{colim}_{i\in\mathcal{I}} X_i$ and $\pi_i\colon \lim_{i\in\mathcal{I}} X_i \to X_i$ are the canonical maps in Top, then

$$\operatorname*{colim}_{i\in\mathcal{I}}(X_i, \mathcal{O}_{X_i}) = (\operatorname*{colim}_{i\in\mathcal{I}} X_i, \lim_{i\in\mathcal{I}}(\iota_i)_*(\mathcal{O}_{X_i})),$$
$$\lim_{i\in\mathcal{I}}(X_i, \mathcal{O}_{X_i}) = (\lim_{i\in\mathcal{I}} X_i, \operatorname*{colim}_{i\in\mathcal{I}}(\pi_i)^*(\mathcal{O}_{X_i})).$$

Proposition 7.49 (Colimits of locally ringed spaces)**.** *The category* LRS_k *admits colimits and the inclusion functor* $\mathrm{LRS}_k \hookrightarrow \mathrm{RS}_k$ *preserves colimits.*

Remark 7.50. The category LRS_k also admits limits, but this is more subtle as the inclusion $\mathrm{LRS}_k \hookrightarrow \mathrm{RS}_k$ does not preserve them. For example, it follows from Theorem 7.55 below that the final object of LRS_k is the underlying space of $\operatorname{Spec}(k)$ (i.e., the prime spectrum of k) equipped with the sheaf of functions.

7.5. **Schemes as locally ringed spaces.** Let X be an algebraic functor. Denote by $\mathcal{O}_X$ the presheaf of functions on $\mathrm{Open}(X) \simeq \mathrm{Open}(|X|)$, given by

$$\mathcal{O}_X(U) = \mathcal{O}(U) = \mathrm{Map}(U, \mathbb{A}^1).$$

By Corollary 4.43, $\mathcal{O}_X$ is then a sheaf of rings on the underlying topological space $|X|$. This construction defines a functor

$$\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{RS}, \quad X \mapsto (|X|, \mathcal{O}_X). \tag{7.51}$$

By Zariski descent for the underlying space (Remark 6.51) and for functions (Corollary 4.43), this functor factors through the Zariski sheafification functor $\mathrm{a}_{\mathrm{Zar}}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}})$.

Warning 7.52. The notation $\mathcal{O}_X$ is used for two different objects associated with the algebraic functor X: it is the unit object in the category Mod_X of quasi-coherent modules over X (Notation 4.20), and it is the sheaf of rings $\mathcal{O}|_{\mathrm{Open}(X)}$ on the underlying space $|X|$. These are two different ways of restricting the functor $\mathcal{O}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^{\mathrm{op}} \to \mathrm{CAlg}$ to X. We will see in §7.6 in which sense these two objects are equivalent when X is a scheme.

[7]It is common to further require C^r-manifolds to be Hausdorff, second-countable, and of bounded dimension; these conditions are equivalent to being embeddable in $(\mathbb{R}^n, \mathcal{C}^r)$ for some n.

Example 7.53 (Prime spectra as locally ringed spaces)**.** Let R be a ring and let $X = \mathrm{Spec}(R)$. Then the ringed space $(|X|, \mathcal{O}_X)$ has the following explicit description:

- The topological space $|X|$ is the prime spectrum $\mathrm{Prim}(R)$.
- Under the equivalence of categories $\mathrm{Sh}(\mathrm{Prim}(R)) \simeq \mathrm{Sh}(\mathrm{Open}^{\mathrm{pr}}(\mathrm{Spec}(R)))$ of Example 6.36, the sheaf $\mathcal{O}_X$ is given by $\mathcal{O}_X(\mathrm{D}(f)) = R_f$.

Given a prime ideal $\mathfrak{p} \in \mathrm{Prim}(R)$, the stalk of $\mathcal{O}_X$ at $\mathfrak{p}$ is the local ring $R_{\mathfrak{p}}$:

$$\mathcal{O}_{X,\mathfrak{p}} = \operatorname*{colim}_{\mathfrak{p}\in|\mathrm{D}(f)|} \mathcal{O}_X(\mathrm{D}(f)) = \operatorname*{colim}_{f\notin\mathfrak{p}} R_f = R_{\mathfrak{p}}.$$

Hence, the ringed space $(\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is in fact a locally ringed space. Moreover, for any ring map $\varphi\colon R \to S$, the induced map $(\mathrm{Prim}(S), \mathcal{O}_{\mathrm{Spec}(S)}) \to (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is a map of locally ringed spaces: for any prime ideal $\mathfrak{q} \in \mathrm{Prim}(S)$, the induced map on stalks is the local map $\varphi_{\mathfrak{q}}\colon R_{\varphi^{-1}(\mathfrak{q})} \to S_{\mathfrak{q}}$.

Recall that both functors

$$|{-}|\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{Top} \quad \text{and} \quad \mathcal{O}\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{CAlg}^{\mathrm{op}}$$

preserve colimits (Proposition 5.24 and Remark 2.52(iv)). Using the explicit description of colimits in RS from Remark 7.48, we deduce that the functor (7.51) is colimit-preserving. By the universal property of presheaves (Theorem 2.9(v)), it is therefore the unique colimit-preserving extension of the functor

$$\mathrm{CAlg}^{\mathrm{op}} \to \mathrm{RS}, \quad R \mapsto (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)}).$$

This functor lands in the (non-full) subcategory LRS $\subset$ RS by Example 7.53. By Proposition 7.49, we deduce that its colimit-preserving extension (7.51) also lands in the subcategory LRS $\subset$ RS. The following proposition summarizes these observations:

Proposition 7.54. *There is a factorization*

$$\begin{array}{ccc} \mathrm{Fun}(\mathrm{CAlg},\mathrm{Set}) & \xrightarrow{(|-|,\mathcal{O})} & \mathrm{RS} \\ \big\downarrow{\scriptstyle \mathrm{a}_{\mathrm{Zar}}} & & \big\uparrow \\ \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{CAlg}^{\mathrm{op}}) & \dashrightarrow & \mathrm{LRS}. \end{array}$$

Theorem 7.55 (The adjunction between rings and locally ringed spaces)**.** *The global section functor*

$$\mathrm{LRS} \to \mathrm{CAlg}^{\mathrm{op}}, \quad (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$$

has a right adjoint given by

$$\mathrm{CAlg}^{\mathrm{op}} \to \mathrm{LRS}, \quad R \mapsto (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)}).$$

Since $\mathcal{O}(\mathrm{Spec}(R)) \simeq R$, the counit of this adjunction is an isomorphism, which implies the following:

Corollary 7.56. *The prime spectrum functor*

$$\mathrm{CAlg}^{\mathrm{op}} \to \mathrm{LRS}, \quad R \mapsto (\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$$

is fully faithful.

The following result follows from Corollary 7.56 by "standard descent arguments":

Theorem 7.57 (Schemes as locally ringed spaces)**.** *Let* $\mathcal{P}\mathrm{rim}$ *be the collection of locally ringed spaces* $(\mathrm{Prim}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ *where* R *is any ring. Then the functor* $(|{-}|, \mathcal{O})\colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set}) \to \mathrm{LRS}$ *restricts to an equivalence of categories*

$$\mathrm{Sch} \xrightarrow{\simeq} \mathrm{LRS}(\mathcal{P}\mathrm{rim}).$$

Remark 7.58. Both the algebraic functor $\mathrm{Spec}(R)$ and the locally ringed space $\mathrm{Prim}(R)$ are universal objects with ring of functions R in their respective categories:

(i) For any algebraic functor X and ring map $\varphi\colon R \to \mathcal{O}(X)$, there is a unique map $X \to \mathrm{Spec}(R)$ inducing φ on rings of functions.

(ii) For any locally ringed space X and ring map $\varphi\colon R \to \mathcal{O}(X)$, there is a unique map $X \to \mathrm{Prim}(R)$ inducing φ on rings of functions.

For $\mathrm{Spec}(R)$, this holds by "abstract nonsense" (Remark 2.54), whereas for $\mathrm{Prim}(R)$ it is the content of Theorem 7.55.

7.6. **Quasi-coherent modules on schemes.** If X is an algebraic functor, the category Mod_X of quasi-coherent modules over X is by definition a limit of a diagram of categories of modules over rings, with transition functors given by extension of scalars (Definition 4.13). For any ring R, the category Mod_R is an abelian category, and it is easy to show that the limit of a diagram of abelian categories and exact functors is again an abelian category. However, extension of scalars is only right exact and not exact in general, so that the category Mod_X is not necessarily abelian.

If X is a scheme, the situation is much better. Indeed, in the category of Zariski sheaves, we can write X as the colimit of a diagram of affine schemes and *open immersions*. If $j\colon \mathrm{Spec}(B) \hookrightarrow \mathrm{Spec}(A)$ is an open immersion of affine schemes, the ring map $A \to B$ is *flat*, so that the functor $j^*\colon \mathrm{Mod}_A \to \mathrm{Mod}_B$ is exact. Hence, using Zariski descent for the category of quasi-coherent modules (Proposition 4.41), we can write Mod_X as the limit of a diagram of abelian categories and exact functors:

$$\mathrm{Mod}_X \xrightarrow{\simeq} \lim_{\substack{\mathrm{Spec}(R)\hookrightarrow X \\ \text{open immersion}}} \mathrm{Mod}_R.$$

This implies that Mod_X is an abelian category, and that for any open immersion $j\colon U \hookrightarrow X$, the functor $j^*\colon \mathrm{Mod}_X \to \mathrm{Mod}_U$ is exact. Thus, for any open covering $(U_i \subset X)_{i\in I}$, the functor $\mathrm{Mod}_X \to \prod_{i\in I}\mathrm{Mod}_{U_i}$ is both exact and conservative, so that it preserves and detects monomorphisms, epimorphisms, and isomorphisms.

Furthermore, while the forgetful functors $\mathrm{Rad}_R \hookrightarrow \mathrm{Id}_R \hookrightarrow (\mathrm{Mod}_R)_{/R}$ are not compatible with arbitrary base change and hence do not extend to algebraic functors, they are compatible with base change along open immersions, so that they do extend to schemes: for any open immersion $\mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$, the squares

$$\begin{array}{ccccc} \mathrm{Rad}_R & \hookrightarrow & \mathrm{Id}_R & \hookrightarrow & (\mathrm{Mod}_R)_{/R} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Rad}_S & \hookrightarrow & \mathrm{Id}_S & \hookrightarrow & (\mathrm{Mod}_S)_{/S} \end{array}$$

commute. These considerations lead to the following results:

Theorem 7.59 (Quasi-coherent modules and ideals on schemes)**.** *Let X be a scheme, let $(U_i \subset X)_{i\in I}$ be an open covering by affine schemes, and let $u_i\colon \mathrm{Spec}(R_i) \simeq U_i \hookrightarrow X$ be the corresponding points.*

(i) *The category Mod_X is abelian.*
(ii) *For any open immersion $j\colon U \hookrightarrow X$, the pullback functor $j^*\colon \mathrm{Mod}_X \to \mathrm{Mod}_U$ is exact.*
(iii) *A map $f\colon M \to N$ in Mod_X is a monomorphism if and only if the maps $f(u_i)\colon M(u_i) \to N(u_i)$ are monomorphisms in Mod_{R_i} for all $i \in I$. The same statement holds for epimorphisms and isomorphisms.*
(iv) *A quasi-coherent module $M \in \mathrm{Mod}_X$ is a vector bundle (resp. a line bundle) if and only if each $M(u_i)$ is a vector space (resp. a line) over R_i.*
(v) *There is an isomorphism of posets*

$$\mathrm{Id}_X \simeq \{\textit{subobjects of } \mathcal{O}_X \textit{ in } \mathrm{Mod}_X\}.$$

(vi) *There is an isomorphism of posets*

$$\mathrm{Rad}_X \simeq \{K \in \mathrm{Id}_X \mid \textit{for all } i \in I, \textit{ the ideal } K(u_i) \subset R_i \textit{ is radical}\}.$$

Finally, we explain how to identify quasi-coherent modules over a scheme X with certain sheaves on the underlying topological space $|X|$.

Definition 7.60 (Quasi-coherent sheaf)**.** Let $(X, \mathcal{O}_X)$ be a ringed space and M an $\mathcal{O}_X$-module in $\mathrm{Sh}(X)$. We say that M is a *quasi-coherent sheaf* on $(X, \mathcal{O}_X)$ if it admits a presentation locally on X, i.e., if for every $x \in X$ there is an open neighborhood U of x and an exact sequence of the form

$$\mathcal{O}_U^{(J)} \to \mathcal{O}_U^{(I)} \to M|_U \to 0,$$

where I and J are sets.

Proposition 7.61 (Quasi-coherent modules as sheaves)**.** *Let X be scheme. Then the functor*

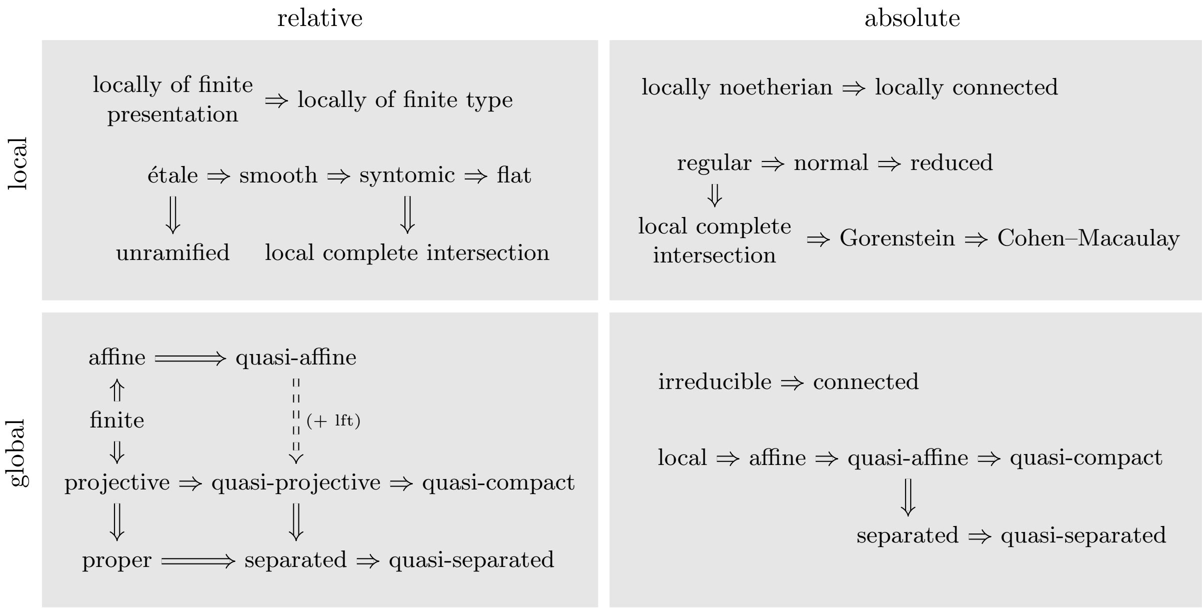
$$\begin{gathered} \mathrm{Mod}_X \to \mathrm{Mod}_{\mathcal{O}_X}(\mathrm{Sh}(|X|)) \\ M \mapsto (U \mapsto \Gamma(U, M)), \end{gathered}$$

is fully faithful and identifies Mod_X with the category of quasi-coherent sheaves on $(|X|, \mathcal{O}_X)$.

8. Properties of schemes

Broadly speaking, there are four different types of properties in geometry. A property is called *local* if it can be checked in a neighborhood of every point, and it is *global* otherwise. In algebraic geometry, local properties are determined by their values on affine schemes and hence come from commutative algebra. For example, as the name implies, "locally of finite presentation" is a local property, which in the affine setting corresponds to the notion of finitely presented algebra (Proposition 2.82). We will see many more examples of local properties of schemes in this chapter. On the other hand, "affine" and "connected" are global properties: every scheme is locally affine, so affineness cannot be checked locally, and connectedness is unrelated to local connectedness. In a different direction, there are *absolute* properties, which are properties of the geometric objects themselves, and *relative* properties, which are properties of "objects over a base", i.e., properties of morphisms. For example, projectivity is a property of S-schemes which heavily depends on the base scheme S, and is thus a relative property: a nonempty projective $\mathbb{C}$-scheme will never be projective as a $\mathbb{Q}$-scheme.

The following table lists a few important examples of all four types of properties in algebraic geometry, as well as some implications between them:



of finite presentation = lfp + qcqs

of finite type = lft + qc

noetherian = locally noetherian + qc

integral = reduced + irreducible

Remark 8.1. Every relative property has an associated absolute property, since every scheme can be regarded as a scheme over $\mathrm{Spec}(\mathbb{Z})$. For example, the absolute versions of "affine", "quasi-affine", "quasi-compact", "quasi-separated", and "separated" are obtained from the relative versions in this way. But for many relative properties, like smoothness and properness, the associated absolute property is not used, since it is very restrictive (e.g., a nonempty $\mathbb{C}$-scheme can never be smooth or proper over $\mathrm{Spec}(\mathbb{Z})$).

8.1. Local properties.

Given a property P of rings, we can consider schemes that admit an open covering by spectra of rings with the property P; this property of schemes is usually called "locally P" or simply "P". This is only well-behaved when the property P is *local* in the following sense:

Definition 8.2 (Local property of rings)**.** A property P of rings is *local* if the following holds:

(i) If R has property P and $f \in R$, then R_f has property P.
(ii) If $(f_1, \ldots, f_n) = R$ and each R_{f_i} has property P, then R has property P.

Definition 8.3 (Local property of schemes)**.** A property P of schemes is *local* if the following holds:

(i) If X has property P and $U \subset X$ is open, then U has property P.
(ii) If $(U_i \subset X)_i$ is an open covering of X and each U_i has property P, then X has property P.

Remark 8.4. Condition (i) of Definition 8.3 says that P defines (for every scheme X) a presheaf of truth values on $\mathrm{Open}(X)$, and Condition (ii) says that this presheaf is a sheaf for the canonical topology.

Proposition 8.5 (Extending absolute local properties)**.** *Every local property of rings extends uniquely to a local property of schemes.*

For properties of morphisms, locality has two aspects: locality on the source and locality on the base:

Definition 8.6. Let P be a property of morphisms of schemes.

(i) P is *local on the source* if, for any morphism $f\colon Y \to X$:
  - If f has property P and $V \subset Y$ is an open subscheme, then $f|V\colon V \to X$ has property P.
  - If $(V_i \subset Y)_{i\in I}$, is an open covering and each $f|V_i\colon V_i \to X$ has property P, then f has property P.

(ii) P is *local on the base* if, for any morphism $f\colon Y \to X$:
  - If f has property P and $U \subset X$ is open, then $f_U\colon Y \times_X U \to U$ has property P.
  - If $(U_i \subset X)_{i\in I}$ is an open covering and each $f_{U_i}\colon Y \times_X U_i \to U_i$ has property P, then f has property P.

(iii) P is *local* if it is both local on the source and local on the base.

(iv) P is *stable under base change* if, for any cartesian square of schemes

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X, \end{array}$$

if f has property P, then f' has property P.

(v) P is *stable under composition* if it contains isomorphism and if $g \circ f$ has property P whenever both f and g have property P.

Definition 8.6 has an obvious analogue for morphisms of rings, and we have the following relative version of Proposition 8.5 (recall from Proposition 4.36(i) that a morphism $Y \to X$ is *affine* if and only if, for every $\mathrm{Spec}(R) \to X$, the pullback $Y \times_X \mathrm{Spec}(R)$ is an affine scheme):

Proposition 8.7 (Extending relative local properties)**.** *Let P be a property of morphisms of rings.*

(i) *If P is local on the base, then it extends uniquely to a property $\hat{P}$ of* affine *morphisms of schemes that is local on the base.*

(ii) *If P is local, then it extends uniquely to a local property $\hat{P}$ of morphisms of schemes.*

In both cases, if P is stable under base change or stable under composition, then so is $\hat{P}$.

Remark 8.8. A property of algebras (i.e., of morphisms of rings) is Zariski-local in the sense of Definition 2.76 if and only if it is local on the *base* in the sense of Definition 8.6.

Remark 8.9. Most relevant properties of morphisms are stable under base change and composition. For such properties, locality on the source implies locality on the base.

Example 8.10 (Local finiteness properties)**.** For morphisms of schemes, the properties "locally of finite presentation" and "locally of finite type" from Definition 2.86 are local. Hence, a morphism of schemes $f\colon Y \to X$ is locally of finite presentation (resp. locally of finite type) if and only if, for every $y \in |Y|$, there exist affine open neighborhoods V of y and U of $f(y)$ with $f(V) \subset U$ such that $\mathcal{O}(V)$ is an $\mathcal{O}(U)$-algebra of finite presentation (resp. of finite type).

Example 8.11 (Flatness)**.** Recall that a map of rings $\varphi\colon R \to S$ is *flat* if the functor $\varphi^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ is exact. This property of morphisms of rings is clearly stable under base change and under composition. It is moreover a local property: in fact, $\varphi\colon R \to S$ is flat if and only if, for every prime ideal $\mathfrak{q} \subset S$, the induced map of local rings $R_{\varphi^{-1}(\mathfrak{q})} \to S_{\mathfrak{q}}$ is flat (this follows immediately from the fact that, for any ring R, the functor $\mathrm{Mod}_R \to \prod_{\mathfrak{p}} \mathrm{Mod}_{R_{\mathfrak{p}}}$ is exact and conservative). Hence, by Proposition 8.7(ii), flatness extends uniquely to a local property of morphisms of schemes, which has the following characterization: $f\colon Y \to X$ is flat if and only if, for every $y \in |Y|$, the map of local rings $f^*\colon \mathcal{O}_{X,f(x)} \to \mathcal{O}_{Y,y}$ is flat.

Example 8.12 (Properties local on the base)**.** The following properties of morphisms of schemes are local on the base, but not on the source:

(i) affine;
(ii) integral;
(iii) finite;
(iv) closed immersion;
(v) open immersion;

(vi) immersion;
(vii) quasi-affine;
(viii) quasi-compact;
(ix) quasi-separated;
(x) separated;
(xi) of finite presentation;
(xii) of finite type.

All these properties are also stable under base change and composition. The first four are properties of affine morphisms and hence arise from Proposition 8.7(i). Properties that are not local on the base are somewhat pathological; these include "projective" and "quasi-projective". Properness is a weakening of projectivity that is local on the base (see §**??**).

8.2. **Reduced schemes.** Recall that a ring R is *reduced* if its nilradical $\mathrm{Nil}(R) = \sqrt{0}$ is the zero ideal. The *reduction* of a ring R is $R_{\mathrm{red}} = R/\mathrm{Nil}(R)$

Definition 8.13 (Reduced scheme)**.** A scheme X is *reduced* if there is an affine open covering $(U_i \subset X)_{i\in I}$ such that each ring $\mathcal{O}(U_i)$ is reduced.

Lemma 8.14. *"Reduced" is a local property of rings.*

By Proposition 8.5, we deduce:

Proposition 8.15. *"Reduced" is a local property of schemes. Moreover,* $\mathrm{Spec}(R)$ *is reduced if and only if R is reduced.*

Remark 8.16.

(i) A scheme X is reduced if and only if, for every point $x \in |X|$, the local ring $\mathcal{O}_{X,x}$ is reduced. The forward implication follows from the fact that localizations of reduced rings are reduced, and the converse from the fact that any ring R is a subring of $\prod_{\mathfrak{p}} R_{\mathfrak{p}}$.
(ii) If X is a reduced scheme, then the ring $\mathcal{O}(X)$ is reduced. Indeed, if $(U_i \subset X)_{i\in I}$ is an open covering of X by reduced affine schemes, then $\mathcal{O}(X)$ is a subring of the reduced ring $\prod_{i\in I} \mathcal{O}(U_i)$.

Example 8.17 (Reduced Proj)**.** Let A be an $\mathbb{N}$-graded ring and let $I \subset A_+$ be a homogeneous subset such that $A_+ \subset \sqrt{(I)}$. Recall that $\mathrm{Proj}(A)$ has an open covering by the affine schemes $\mathrm{Spec}(A_{(f)})$ for $f \in I$. Hence, $\mathrm{Proj}(A)$ is reduced if and only if each ring $A_{(f)}$ with $f \in I$ is reduced. A sufficient condition is that A itself be reduced. For example, if R is reduced and $n \geq -1$, then $\mathbb{P}^n_R$ is reduced.

If X is a scheme, recall that there is an embedding

$$\mathrm{Rad}_X \hookrightarrow \mathrm{Id}_X$$

(Theorem 7.59(vi)), so that "radical" is a meaningful property of quasi-coherent ideals over X. Concretely, if $K \in \mathrm{Id}_X$ and $(x_i \colon \mathrm{Spec}(R_i) \hookrightarrow X)_{i\in I}$ is an affine open covering of X, then K is radical if and only if each ideal $K(x_i) \in \mathrm{Id}_{R_i}$ is radical.

Proposition 8.18 (Classification of reduced closed subschemes)**.** *Let X be a scheme. A closed subscheme $Z \subset X$ is reduced if and only if the quasi-coherent ideal I over X such that* $\mathrm{V}(I) = Z$ *is radical.*

Corollary 8.19. *Let X be a scheme. Then the composite*

$$\mathrm{Closed}^{\mathrm{red}}(X) \hookrightarrow \mathrm{Closed}(X) \xrightarrow{|-|} \mathrm{Closed}(|X|)$$

is bijective. In particular, the second map is surjective.

The following commutative diagram of posets summarizes the classification of closed and open subschemes of a scheme X, where "c" means taking the complement (see Proposition 4.23, Proposition 5.24, Remark 5.29, Proposition 8.18):

$$\begin{array}{ccccc}
 & \overset{\sim}{\nearrow} & \mathrm{Closed}(|X|)^{\mathrm{op}} & \xrightarrow[\sim]{\mathrm{c}} & \mathrm{Open}(|X|) \\
 & & \big\uparrow\!\!\uparrow{\scriptstyle |-|} & & \wr\big\uparrow{\scriptstyle |-|} \\
\mathrm{Closed}^{\mathrm{red}}(X)^{\mathrm{op}} & \hookrightarrow & \mathrm{Closed}(X)^{\mathrm{op}} & \xrightarrow{\mathrm{c}} & \mathrm{Open}(X) \\
\wr\big\uparrow & & \wr\big\uparrow{\scriptstyle \mathrm{V}} & & \wr\big\uparrow{\scriptstyle \mathrm{D}} \\
\mathrm{Rad}_X & \hookrightarrow & \mathrm{Id}_X & \xrightarrow{\sqrt{\ }} & \mathrm{Rad}_X.
\end{array}$$

Example 8.20 (Reduced closed subschemes of Proj)**.** Let A be an $\mathbb{N}$-graded ring. Recall that the closed subsets of $|\mathrm{Proj}(A)|$ are in bijection with the saturated radical homogeneous ideals $I \subset A$ (Remark 3.82). By Corollary 8.19 and Example 8.17, we obtain a bijection

$$\{\text{saturated radical homogeneous ideals in } A\} \xrightarrow{\sim} \mathrm{Closed}^{\mathrm{red}}(X), \quad I \mapsto \mathrm{V}(I).$$

Definition 8.21 (Reduction)**.** Let X be a scheme. The *reduction* of X is the closed subscheme

$$X_{\mathrm{red}} = \mathrm{V}(\sqrt{0}) \subset X.$$

Proposition 8.22. *Let* $\mathrm{Sch}^{\mathrm{red}} \subset \mathrm{Sch}$ *be the full subcategory of reduced schemes. Then the inclusion* $\mathrm{Sch}^{\mathrm{red}} \hookrightarrow \mathrm{Sch}$ *has a right adjoint given by* $X \mapsto X_{\mathrm{red}}$*.*

Remark 8.23. Let X be a scheme. Then the closed immersion $X_{\mathrm{red}} \hookrightarrow X$ induces an isomorphism on underlying spaces. Indeed, the open complement of $X_{\mathrm{red}} = \mathrm{V}(\sqrt{0})$ in X is the empty scheme $\mathrm{D}(\sqrt{0}) = \varnothing$. Hence, as a locally ringed space, X_{red} has the form $(|X|, \mathcal{O}_X^{\mathrm{red}})$, where $\mathcal{O}_X^{\mathrm{red}}$ is given by $U \mapsto \mathcal{O}(U_{\mathrm{red}})$. Note that the sheaf $\mathcal{O}_X^{\mathrm{red}}$ is the sheafification of the presheaf of rings $U \mapsto \mathcal{O}_X(U)_{\mathrm{red}}$, since the canonical map $\mathcal{O}_X(-)_{\mathrm{red}} \to \mathcal{O}_X^{\mathrm{red}}$ is an isomorphism on affine opens.

8.3. **Noetherian schemes.** Recall that a ring R is *noetherian* if every ascending chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ in R stabilizes. Equivalently, R is noetherian if every descending sequence of closed subschemes $Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ in $\mathrm{Spec}(R)$ stabilizes.

Proposition 8.24 (Characterizations of noetherian rings)**.** *For a ring R, the following conditions are equivalent:*

(i) *R is noetherian.*
(ii) *Every submodule of a finitely generated R-module is finitely generated.*
(iii) $\mathrm{Mod}_R^{\mathrm{ft}}$ *is an abelian category such that the inclusion* $\mathrm{Mod}_R^{\mathrm{ft}} \hookrightarrow \mathrm{Mod}_R$ *is exact.*
(iv) *Every R-module of finite type is of finite presentation:* $\mathrm{Mod}_R^{\mathrm{ft}} = \mathrm{Mod}_R^{\mathrm{fp}}$.
(v) *Every R-algebra of finite type is of finite presentation:* $\mathrm{CAlg}_R^{\mathrm{ft}} = \mathrm{CAlg}_R^{\mathrm{fp}}$.

Remark 8.25. Condition (v) in Proposition 8.24 implies that, if R is noetherian, then any finitely generated R-algebra is again noetherian. The fact that $R[x_1, \dots, x_n]$ is noetherian is Hilbert's basis theorem.

Definition 8.26 (Locally noetherian, noetherian scheme)**.** A scheme X is *locally noetherian* if there is an affine open covering $(U_i \subset X)_{i \in I}$ such that each ring $\mathcal{O}(U_i)$ is noetherian. A scheme is *noetherian* if it is locally noetherian and quasi-compact.

Lemma 8.27. *"Noetherian" is a local property of rings.*

By Proposition 8.5, we deduce:

Proposition 8.28. *"Locally noetherian" is a local property of schemes. Moreover,* $\mathrm{Spec}(R)$ *is locally noetherian if and only if R is noetherian.*

Example 8.29 (Noetherian Proj)**.** Let A be an $\mathbb{N}$-graded ring and let $I \subset A_+$ be a homogeneous subset such that $A_+ \subset \sqrt{(I)}$. Then $\mathrm{Proj}(A)$ is locally noetherian if and only if each ring $A_{(f)}$ with $f \in I$ is noetherian. It is moreover quasi-compact if and only if a finite such set I exists. In particular, if A itself is noetherian, then A_+ is a finitely generated ideal and hence $\mathrm{Proj}(A)$ is noetherian. For example, if R is noetherian and $n \geq -1$, then $\mathbb{P}^n_R$ is noetherian.

Remark 8.30 (Coherent rings)**.** A ring R is called *coherent* if $\mathrm{Mod}_R^{\mathrm{fp}}$ is an abelian category such that the inclusion $\mathrm{Mod}_R^{\mathrm{fp}} \hookrightarrow \mathrm{Mod}_R$ is exact. By Proposition 8.24, noetherian rings are coherent, but the converse fails. For example, any valuation ring is coherent, and if R is noetherian and I is any set, then the polynomial ring $R[x_i \mid i \in I]$ is coherent. One can show that "coherent" is also a local property of rings, so that it extends uniquely to a local property of schemes.

Proposition 8.31. *A scheme X is noetherian if and only if every descending sequence of closed subschemes $Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ in X stabilizes.*

Definition 8.32 (Noetherian space)**.** A topological space T is *noetherian* if every descending sequence of closed subsets $Z_0 \supset Z_1 \supset Z_2 \supset \cdots$ in T stabilizes.

Remark 8.33. If T is a noetherian space, then T is quasi-compact. Moreover, any subspace of T is again noetherian, and hence quasi-compact.

Remark 8.34. By Proposition 8.31 and Corollary 8.19, if X is a noetherian scheme, then the underlying space $|X|$ is a noetherian space. However, the converse is not true. For example, if k is a field, then $R = k[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$ is a non-noetherian ring such that $|\mathrm{Spec}(R)|$ is a single point (since $R_{\mathrm{red}} = k$).

Remark 8.35 (Noetherian induction)**.** To show that all closed subsets of a noetherian space have a certain property P, it suffices to show that Z has property P whenever all $W \subsetneq Z$ have property P. The same holds for closed subschemes of a noetherian scheme (by Proposition 8.31). This method of proof is called *noetherian induction.*

Just like for noetherian rings, there is no difference between "finite presentation" and "finite type" for noetherian schemes:

Proposition 8.36.

(i) *A locally noetherian scheme is quasi-separated*

(ii) *If X is locally noetherian and $f\colon Y \to X$ is locally of finite type, then Y is locally noetherian and f is locally of finite presentation.*

(iii) *If X is noetherian and $f\colon Y \to X$ is of finite type, then Y is noetherian and f is of finite presentation.*

8.4. **Irreducible components.** Recall that a topological space T is *irreducible* if it is nonempty and for every nonempty open subsets $U, V \subset T$, the intersection $U \cap V$ is nonempty. If T is sober (e.g., the underlying space of a scheme), this is equivalent to the existence of a *generic point*, i.e., a point $\eta \in T$ such that $\overline{\{\eta\}} = T$ (Proposition 5.13). According to Definition 5.30, a scheme X is *irreducible* if its underlying space $|X|$ is irreducible.

Definition 8.37 (Irreducible components)**.**

(i) The *irreducible components* of a topological space T are the maximal irreducible subsets of T.

(ii) The *irreducible components* of a scheme X are the irreducible components of the underlying space $|X|$.

Remark 8.38. Let T be a topological space.

(i) By Zorn's lemma, every irreducible subset of T is contained in an irreducible component of T. In particular, since points are irreducible, T is the union of its irreducible components.

(ii) A subset $A \subset T$ is irreducible if and only if its closure $\bar{A} \subset T$ is. It follows that the irreducible components of T are closed.

Example 8.39 (Irreducible components of affine schemes)**.** Let R be a ring. Under the bijection

$$\begin{aligned} \mathrm{Rad}_R &\xrightarrow{\sim} \{\text{closed subsets of } \mathrm{Prim}(R)\}, \\ I &\mapsto \{\mathfrak{p} \mid I \subset \mathfrak{p}\}, \end{aligned}$$

the irreducible closed subsets of $\mathrm{Prim}(R)$ correspond to the prime ideals of R. Hence, the irreducible components of $\mathrm{Spec}(R)$ correspond to the minimal prime ideals of R. In particular, $\mathrm{Spec}(R)$ is irreducible if and only if R has a unique minimal prime ideal. Since the intersection of all prime ideals is always the nilradical, this is equivalent to the nilradical $\mathrm{Nil}(R)$ being prime.

Example 8.40 (Irreducible components of Proj)**.** Let A be an $\mathbb{N}$-graded ring. Recall that the closed subsets of $|\mathrm{Proj}(A)|$ are in bijection with the saturated radical homogeneous ideals $I \subset A$ (Remark 3.82). Under this bijection, the irreducible closed subsets correspond to the saturated homogeneous prime ideals, and hence the irreducible components correspond to the minimal such ideals. In particular, $\mathrm{Proj}(A)$ is irreducible if and only if $\mathrm{Nil}(A)^{\mathrm{sat}}$ is prime.

Example 8.41. Let k be a field.

(i) The affine k-scheme $\mathrm{Spec}(k[x,y]/(xy))$ has two irreducible components, corresponding to the minimal prime ideals (x) and (y) in $k[x,y]/(xy)$. They meet in a single point, corresponding to the maximal ideal (x,y).

(ii) The projective k-scheme $\mathrm{Proj}(k[x,y,z]/(xyz))$ has three irreducible components, corresponding to the minimal saturated prime ideals (x), (y), and (z) in $k[x,y,z]/(xyz)$. These components are in a triangle configuration, each pair of components meeting in a single point away from the third component.

Proposition 8.42 (Irreducible components of noetherian spaces)**.** *A noetherian topological space has finitely many irreducible components. In particular, a noetherian scheme has finitely many irreducible components.*

Definition 8.43 (Integral scheme, function field)**.** A scheme X is *integral* if it is irreducible and reduced. The *function field* of X (or *field of rational functions* on X) is the residue field $\kappa(\eta)$ of the generic point $\eta \in |X|$.

Example 8.44 (Integral affine schemes)**.** Let R be a ring. Then $\mathrm{Spec}(R)$ is integral if and only if R is an integral domain (in which case the function field of $\mathrm{Spec}(R)$ is the fraction field $\mathrm{Frac}(R)$). This follows from Proposition 8.15 and Remark 8.39.

Proposition 8.45. *Let X be an integral scheme with generic point η.*

(i) *For every nonempty open subscheme $U \subset X$, the map $\mathcal{O}(U) \to \kappa(\eta)$ is injective.*
(ii) *For every nonempty affine open subscheme $U \subset X$, the map $\mathcal{O}(U) \to \kappa(\eta)$ exhibits $\kappa(\eta)$ as the fraction field of $\mathcal{O}(U)$.*
(iii) *For every point $x \in |X|$, the map $\mathcal{O}_{X,x} \to \kappa(\eta)$ exhibits $\kappa(\eta)$ as the fraction field of $\mathcal{O}_{X,x}$.*

Example 8.46 (Integral Proj)**.** If A is an integral $\mathbb{N}$-graded ring such that $A_+ \neq 0$, then $\mathrm{Proj}(A)$ is an integral scheme: this follows from Examples 8.17 and 8.40 (the assumption that $A_+ \neq 0$ ensures that $(0)^{\mathrm{sat}} = (0)$). By Proposition 8.45(ii), the function field of $\mathrm{Proj}(A)$ is then the fraction field of $A_{(f)}$ for any nonzero homogeneous element $f \in A_+$, i.e., it is $\mathrm{Frac}(A)_0$. For example, if R is an integral domain with fraction field K and $n \geq 0$, then $\mathbb{P}^n_R$ is an integral scheme with function field $K(x_1, \dots, x_n)$.

Remark 8.47 (Rational function)**.** The notion of rational function can be defined for any scheme X as follows. The *sheaf of rational functions* $\mathcal{K}_X$ on $|X|$ is the sheafification of the presheaf of rings $U \mapsto \mathcal{O}(U)[S_U^{-1}]$, where $S_U \subset \mathcal{O}(U)$ is the multiplicative subset of all functions f such that $f|V$ is not a zero divisor in $\mathcal{O}(V)$ for all opens $V \subset U$. Since sheafification is left exact, there is a canonical monomorphism of sheaves $\mathcal{O}_X \hookrightarrow \mathcal{K}_X$. If X is integral with generic point η, then $\mathcal{K}_X$ is the constant sheaf with $\mathcal{K}_X(X) = \kappa(\eta)$. If X is locally noetherian, one can show that

$$\mathcal{K}_X(X) = \operatorname*{colim}_{U \subset X} \mathcal{O}(U),$$

where the colimit ranges over the open subschemes $U \subset X$ whose closure is X (Definition 3.86).

8.5. **Dimension theory.** We review the basics of dimension theory. Define the *length* of a poset P to be the supremum of the lengths of chains in P:

$$\mathrm{length}(P) = \sup\{n \in \mathbb{N} \mid \text{there is a chain } x_0 \subsetneqq \cdots \subsetneqq x_n \text{ in } P\} \in \mathbb{N} \cup \{\pm\infty\}.$$

Definition 8.48 (Krull dimension, codimension)**.** Let T be a topological space.

(i) The *Krull dimension* of T, denoted by $\dim(T)$, is the length of the poset of closed irreducible subsets of T.
(ii) Given an irreducible subset $K \subset T$, the *codimension* of K in T, denoted by $\mathrm{codim}(K, T)$, is the length of the poset of closed irreducible subsets of T containing K.

If X is a scheme, we also apply these definitions to the underlying space of X

Remark 8.49 (Specialization)**.** If T is a topological space and $x, y \in T$, we say that x *specializes* to y, and we write $x \rightsquigarrow y$, if $y \in \overline{\{x\}}$. This defines a partial order on the underlying set of T. If T is sober, there is an isomorphism of posets

$$(T, \rightsquigarrow) \xrightarrow{\sim} (\{\text{irreducible closed subsets of } T\}, \supset), \quad x \mapsto \overline{\{x\}},$$

by Proposition 5.13. We can therefore rephrase Definition 8.48 in terms of the poset of points of T. For example,

$$\dim(T) = \mathrm{length}(T, \rightsquigarrow).$$

Remark 8.50. Let T be a topological space.

(i) By definition, we have

$$\dim(T) = \sup_K \mathrm{codim}(K, T),$$

where K ranges over the irreducible closed subsets of T. If T is sober, we equivalently have

$$\dim(T) = \sup_{x \in T} \mathrm{codim}(x, T).$$

(ii) If $(T_i \subset T)_{i \in I}$ is either an open covering of T or a closed covering of T, then

$$\dim(T) = \sup_{i \in I} \dim(T_i).$$

Definition 8.51 (Krull dimension of schemes and rings)**.**

(i) If X is a scheme, the *Krull dimension* of X is the Krull dimension of its underlying space:

$$\dim(X) = \dim(|X|).$$

We similarly define $\operatorname{codim}(K, X)$ for K an irreducible subset of $|X|$ or subscheme of X.

(ii) If R is a ring, the *Krull dimension* of R is

$$\dim(R) = \dim(\operatorname{Spec}(R)) = \dim(\operatorname{Prim}(R)) = \operatorname{length}(\operatorname{Prim}(R), \subset).$$

Remark 8.52. Let X be a scheme. For any $x \in |X|$ and any open neighborhood U of x, we have

$$\operatorname{codim}(x, X) = \operatorname{codim}(x, U) = \dim(\mathcal{O}_{X,x}).$$

Indeed, every point in X specializing to x lies in U, so the first two numbers are the lengths of the same poset, and taking U to be an affine, we see that this poset is isomorphic to the poset of prime ideals in $\mathcal{O}_{X,x}$ (since the prime ideals in $R_\mathfrak{p}$ correspond to the prime ideals contained in $\mathfrak{p}$). By Remark 8.50(i), we deduce that

$$\dim(X) = \sup_{x \in X} \dim(\mathcal{O}_{X,x}).$$

Because of this, local rings play a central role in dimension theory.

Example 8.53. If R is a principal ideal domain, then

$$\dim(R) = \begin{cases} 0 & \text{if } R \text{ is a field,} \\ 1 & \text{otherwise.} \end{cases}$$

Hence, if X is a scheme all of whose local rings are principal ideal domains, then $\dim(X) \leq 1$.

Theorem 8.54. *Let R be a noetherian ring and let $X = \operatorname{Spec}(R)$.*

(i) (Krull's principal ideal theorem) *Let $f \in R$ and let K be an irreducible component of* $\mathrm{V}(f) \subset X$. *Then*

$$\operatorname{codim}(K, X) = \begin{cases} 1 & \text{if } f \text{ is not a zero divisor in } R_{\mathrm{red}}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) (Krull's height theorem) *Let $f_1, \dots, f_n \in R$ and let $Y = \mathrm{V}(f_1, \dots, f_n) \subset X$. For every irreducible subset $K \subset |Y|$,*

$$\operatorname{codim}(K, X) \leq \operatorname{codim}(K, Y) + n.$$

(iii) *Let $K \subset |X|$ be an irreducible closed subset with* $\operatorname{codim}(K, X) = n$. *Then there exist $f_1, \dots, f_n \in R$ such that K is an irreducible component of* $\mathrm{V}(f_1, \dots, f_n) \subset X$.

Corollary 8.55. *Let R be a noetherian ring and let $\mathfrak{p} \subset R$ be a prime ideal. Then*

$$\operatorname{codim}(\mathrm{V}(\mathfrak{p}), \operatorname{Spec}(R)) \leq \min\left\{ n \in \mathbb{N} \;\middle|\; \text{there exists } f_1, \dots, f_n \in \mathfrak{p} \text{ with } \mathfrak{p} = \sqrt{(f_1, \dots, f_n)} \right\} < \infty.$$

If $f \colon Y \to X$ is a morphism of schemes and $x \in |X|$, the *fiber* of f at x is the scheme

$$Y_x = \operatorname{Spec}(\kappa(x)) \times_X Y.$$

Since $\operatorname{Spec}(\kappa(x)) \hookrightarrow X$ is a monomorphism (Remark 7.4), we have a canonical homeomorphism $|Y_x| \xrightarrow{\sim} \{x\} \times_{|X|} |Y|$ (Proposition 5.27). The following result is a consequence of Theorem 8.54(ii,iii):

Proposition 8.56. *Let $f \colon Y \to X$ be a morphism of schemes, let $y \in |Y|$ and let $x = f(y) \in |X|$.*

(i) *If $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are noetherian, then*

$$\operatorname{codim}(y, Y) \leq \operatorname{codim}(x, X) + \operatorname{codim}(y, Y_x).$$

(ii) *If $f^* \colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat, then*

$$\operatorname{codim}(y, Y) \geq \operatorname{codim}(x, X) + \operatorname{codim}(y, Y_x).$$

Applying Proposition 8.56 to the projection $X \times \mathbb{A}^1 \to \mathbb{A}^1$ and using the fact that $\dim(k[x]) = 1$ if k is a field, we get the following result:

Corollary 8.57. *If X is a locally noetherian scheme, then* $\dim(X \times \mathbb{A}^1) = \dim(X) + 1$.

Example 8.58 (Affine and projective spaces)**.** Let X be a locally noetherian scheme and let V be a vector bundle of rank $n \geq 1$ over X. Then

$$\dim(\mathbb{A}(V)) = \dim(X) + n \quad \text{and} \quad \dim(\mathbb{P}(V)) = \dim(X) + n - 1$$

Indeed, as both statements are local on X by Remark 8.50(ii), we can assume that V is free, so that $\mathbb{A}(V) \simeq X \times \mathbb{A}^n$ and $\mathbb{P}(V) \simeq X \times \mathbb{P}^{n-1}$. Since $\mathbb{P}^{n-1}$ has an open covering by copies of $\mathbb{A}^{n-1}$, the claims now follow from Corollary 8.57.

Theorem 8.59 (Dimension of schemes of finite type over a field)**.** *Let k be a field and let X be an integral k-scheme locally of finite type with generic point η.*

(i) *The Krull dimension of X equals the transcendence degree of the function field $\kappa(\eta)$ over k.*
(ii) *For any irreducible closed subscheme $K \subset X$, we have $\dim(X) = \dim(K) + \operatorname{codim}(K, X)$.*

Warning 8.60. Theorem 8.59(ii) does not hold more generally for locally noetherian integral schemes. For example, let $X = \operatorname{Spec}(\mathbb{Z}_{(p)}[x])$ and $K = \mathrm{V}(px - 1)$. Then X has dimension 2 (by Example 8.53 and Corollary 8.57), but $K \simeq \operatorname{Spec}(\mathbb{Q})$ has dimension zero.

8.6. **Regular schemes.** Combining Theorem 8.54(ii) and Corollary 8.55, we obtain the following characterization of the dimension of noetherian local rings:

Proposition 8.61 (Dimension of noetherian local rings)**.** *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Then*

$$\dim(R) = \min\Big\{n \in \mathbb{N} \mid \textit{there exists } f_1, \dots, f_n \in \mathfrak{m} \textit{ with } \mathfrak{m} = \sqrt{(f_1, \dots, f_n)}\Big\} < \infty.$$

Definition 8.62 (System of parameters)**.** Let R be a noetherian local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. A d-tuple $(f_1, \dots, f_d)$ such that $\mathfrak{m} = \sqrt{(f_1, \dots, f_d)}$ is called a *system of parameters* of R. By Proposition 8.61, every noetherian local ring admits a system of parameters.

The following result is a direct consequence of Proposition 8.61:

Corollary 8.63 (Quotients of noetherian local rings)**.** *Let R be a noetherian local ring of Krull dimension d and let $f_1, \dots, f_n \in \mathfrak{m}$. The following assertions are equivalent:*

(i) *The local ring $R/(f_1, \dots, f_n)$ has Krull dimension $d - n$.*
(ii) *The sequence $(f_1, \dots, f_n)$ can be completed to a system of parameters of R.*

Using Proposition 8.61 and Theorem 8.54(i) inductively, we deduce:

Corollary 8.64. *Let R be a noetherian local ring and let $(f_1, \dots, f_n)$ be a regular sequence in $\mathfrak{m}$, i.e., such that f_i is not a zero divisor in $R/(f_1, \dots, f_{i-1})$ for each i. Then*

$$\dim(R/(f_1, \dots, f_n)) = \dim(R) - n.$$

Definition 8.65 (Depth)**.** Let R be a noetherian local ring. The *depth* of R is the maximal length of a regular sequence in $\mathfrak{m}$. By Corollary 8.64, we have $\operatorname{depth}(R) \leq \dim(R)$.

Definition 8.66 (Cohen–Macauley, regular local ring)**.** Let R be a noetherian local ring.

(i) R is called *Cohen–Macauley* if $\operatorname{depth}(R) = \dim(R)$, i.e., if $\mathfrak{m}$ contains a regular sequence of length $\dim(R)$.
(ii) R is called *regular* if it admits a system of parameters generating $\mathfrak{m}$, i.e., if $\mathfrak{m}$ can be generated by $\dim(R)$ elements.

Proposition 8.67 (Regular $\Rightarrow$ Cohen–Macauley)**.** *Let R be a regular local ring of Krull dimension d. If $\mathfrak{m} = (f_1, \dots, f_d)$, then $(f_1, \dots, f_d)$ is a regular sequence. In particular, R is Cohen–Macauley.*

Example 8.68 (The coordinate axes in the plane)**.** Let k be a field and let R be the local ring of $\operatorname{Spec}(k[x, y]/(xy)) \subset \mathbb{A}^2_k$ at the origin, with maximal ideal $\mathfrak{m} = (x, y)$. Then

$$\sqrt{(0)} \neq \mathfrak{m} = \sqrt{(x + y)},$$

so that $x + y$ is a system of parameters of R and $\dim(R) = 1$. As $x + y$ is not a zero divisor in R, we also have $\operatorname{depth}(R) = 1$, so that R is Cohen–Macauley. However, $\mathfrak{m}$ cannot be generated by fewer than two elements, so that R is not regular.

Example 8.69 (Regular local rings of dimension ≤ 1)**.** Let R be a noetherian local ring. Then R is regular of Krull dimension 0 if and only if it is a field, as $\mathfrak{m}$ must be the zero ideal, and it is regular of Krull dimension 1 if and only if it is a discrete valuation ring, i.e., a noetherian ring in which divisibility is a total order (or equivalently a local principal ideal domain) that is not a field. Indeed, if R is principal and not a field, then it is regular of dimension 1 by definition. Conversely, if R is regular of dimension 1, then $\mathfrak{m} = (f)$ for some $f \neq 0$. Since $\bigcap_{n\in\mathbb{N}} \mathfrak{m}^n = 0$ by Krull's intersection theorem, every element of R has the form uf^n with $n \in \mathbb{N}$ and $u \in R^\times$, so that R is principal.

Definition 8.70 (Zariski (co)tangent space)**.** Let $(X, \mathcal{O}_X)$ be a locally ringed space and let $x \in X$.

(i) The *Zariski cotangent space* at x is the $\kappa(x)$-module $\mathrm{T}^*_x(X) = \mathfrak{m}_x/\mathfrak{m}_x^2$.
(ii) The *Zariski tangent space* at x is the $\kappa(x)$-module $\mathrm{T}_x(X) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.

Example 8.71 (Tangent spaces in differential geometry)**.** If the locally ringed space $(X, \mathcal{O}_X)$ is a C^∞-manifold (see Example 7.47), then there is a surjective map

$$\{\mathrm{C}^\infty\text{-maps } \gamma\colon (-1,1) \to X \text{ such that } \gamma(0) = x\} \twoheadrightarrow \mathrm{T}_x(X), \quad \gamma \mapsto ([f] \mapsto (f \circ \gamma)'(0)),$$

which allows us to identify tangent vectors at x with equivalence classes of curves through x.

Remark 8.72 (Regularity via the cotangent space)**.** Let R be a noetherian local ring. The Zariski cotangent space of $\mathrm{Spec}(R)$ at the closed point is the $\kappa(\mathfrak{m})$-module $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{m} \otimes_R \kappa(\mathfrak{m})$. By Nakayama's lemma, we have

$$\dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2) = \min\{n \in \mathbb{N} \mid \text{there exists } f_1, \ldots, f_n \in \mathfrak{m} \text{ with } \mathfrak{m} = (f_1, \ldots, f_n)\}.$$

By Proposition 8.61, we therefore have $\dim(R) \leq \dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2)$. Moreover, R is regular if and only if $\dim(R) = \dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2)$.

From Remark 8.72 and Nakayama's lemma, we get the following useful criterion:

Corollary 8.73 (Quotients of regular local rings)**.** *Let R be a regular local ring of Krull dimension d and let $f_1, \ldots, f_n \in \mathfrak{m}$. The following assertions are equivalent:*

(i) *The local ring $R/(f_1, \ldots, f_n)$ is regular of Krull dimension $d - n$.*
(ii) *The images of $f_1, \ldots, f_n$ in the Zariski cotangent space $\mathfrak{m}/\mathfrak{m}^2$ are $\kappa(\mathfrak{m})$-linearly independent.*

Example 8.74 (The cuspidal cubic)**.** Let k be a field and let $X = \mathrm{Spec}(k[x,y]/(y^2 - x^3))$. We have $\dim(X) = 1$ (for example by Theorem 8.59(i)). The Zariski cotangent space at a point $(a,b) \in X(k)$ is then

$$\mathrm{T}^*_{(a,b)}(X) = (x-a, y-b)/(x-a, y-b)^2 \in \mathrm{Mod}_k,$$

where the ideals are in the ring $k[x,y]/(y^2 - x^3)$. A straightforward computation shows that

$$\mathrm{T}^*_{(a,b)}(X) \simeq (k \oplus k)/(2b, -3a).$$

This has dimension 2 if $(a,b) = (0,0)$ and dimension 1 otherwise, so that the origin is the only singular k-point of X.

Theorem 8.75 (Properties of regular local rings)**.** *Let R be a regular local ring. Then:*

(i) *The canonical map of $\mathbb{N}$-graded $\kappa(\mathfrak{m})$-algebras*

$$\mathrm{Sym}_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2) \to \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i\in\mathbb{N}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is an isomorphism.
(ii) (Auslander–Buchsbaum theorem) *R is a unique factorization domain.*
(iii) *For every prime ideal $\mathfrak{p} \subset R$, $R_\mathfrak{p}$ is a regular local ring.*

By Theorem 8.75(iii), the following definition is compatible with the existing one for local rings:

Definition 8.76 (Regular ring, regular scheme)**.**

(i) A ring R is called *regular* if it is noetherian and, for every prime ideal $\mathfrak{p} \subset R$, $R_\mathfrak{p}$ is a regular local ring.
(ii) A scheme X is called *regular* if it is locally noetherian and, for every point $x \in |X|$, the local ring $\mathcal{O}_{X,x}$ is a regular local ring.

From these definitions and Proposition 8.28, we immediately deduce:

Proposition 8.77. *"Regular" is a local property of schemes. Moreover,* $\mathrm{Spec}(R)$ *is regular if and only if R is regular.*

For completeness, let us mention the following module-theoretic characterization of regular rings, which is the key ingredient in the proof of Theorem 8.75(iii):

Theorem 8.78 (Serre's homological characterization of regularity)**.** *Let R be a noetherian ring and let $n \in \mathbb{N} \cup \{\infty\}$. The following conditions are equivalent:*

(i) *R is regular of Krull dimension $\leq n$.*
(ii) *Every R-module admits a finite projective resolution of length $\leq n$.*
(iii) *For every ideal $I \subset R$, the R-module R/I admits a finite projective resolution of length $\leq n$.*

8.7. **Normal schemes.** Let R be a ring and A an R-algebra. Recall that an element $a \in A$ is *integral* over R if it satisfies a monic polynomial equation with coefficients in R, or equivalent if the R-subalgebra of A generated by a is finite over R. The set of all such elements is an R-subalgebra of A, called the *integral closure* of R in A. An integral domain R is *integrally closed* if it equals its integral closure in its fraction field $\mathrm{Frac}(R)$.

Example 8.79. By Gauss's lemma on primitive polynomials, unique factorization domains are integrally closed. In particular, by the Auslander–Buchsbaum theorem (Theorem 8.75(ii)), regular local rings are integrally closed domains.

Proposition 8.80. *Let R be an integral domain.*

(i) *If R is integrally closed, then $R[S^{-1}]$ is integrally closed for any subset $S \subset R - \{0\}$.*
(ii) *If $R_\mathfrak{m}$ is integrally closed for every maximal ideal $\mathfrak{m} \subset R$, then R is integrally closed.*

Definition 8.81 (Normal ring, normal scheme)**.**

(i) A ring R is called *normal* if, for every prime ideal $\mathfrak{p} \subset R$, $R_\mathfrak{p}$ is an integrally closed domain.
(ii) A scheme X is called *normal* if, for every point $x \in |X|$, $\mathcal{O}_{X,x}$ is an integrally closed domain.

The following result follows immediately from the definitions:

Proposition 8.82. *"Normal" is a local property of schemes. Moreover,* $\mathrm{Spec}(R)$ *is normal if and only if R is normal.*

Remark 8.83. By Proposition 8.80, an integral domain is normal if and only if it is integrally closed.

Remark 8.84 (Schemes whose local rings are domains)**.** For a scheme X, the following conditions are equivalent (cf. Remark 8.16(i)):

(i) For every $x \in |X|$, $\mathcal{O}_{X,x}$ is an integral domain.
(ii) X is reduced and its irreducible components are pairwise disjoint.

In general, this does not imply that X is a coproduct of integral schemes, but this is true if X is locally noetherian, by Proposition 8.42 (if a topological space has finitely many irreducible components that are pairwise disjoint, then they are clopen). In particular, every locally noetherian normal scheme is a coproduct of integral normal schemes.

Construction 8.85 (Normalization)**.** Let X be an integral scheme. The *normalization* of X is a morphism $X^\nu \to X$, which is uniquely determined by the following properties:

(i) X^ν is a normal integral scheme;
(ii) $X^\nu \to X$ is integral;
(iii) $X^\nu \to X$ induces an isomorphism of function fields.

By (ii), X^ν must be $\mathrm{Spec}(A)$ for some integral quasi-coherent algebra A over X. To construct it, recall from §7.6 that

$$\mathrm{CAlg}_X \xrightarrow{\sim} \lim_{\substack{\mathrm{Spec}(R) \hookrightarrow X \\ \text{open immersion}}} \mathrm{CAlg}_R.$$

Let K be the function field of X. For any open immersion $u\colon \mathrm{Spec}(R) \hookrightarrow X$ with $R \neq 0$, R is a domain with fraction field K (Proposition 8.45(ii)). Define $A(u)$ to be 0 if $R = 0$ and the integral closure of R in K otherwise. Using Proposition 8.80(i), we see that for any open immersion $j\colon \mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$, the canonical map $A(u) \otimes_R S \to A(u \circ j)$ is an isomorphism. The algebras $A(u)$ and these canonical isomorphisms form a quasi-coherent algebra A over X with the desired properties.

Example 8.86 (Normalization of the nodal cubic)**.** Let $X = \mathrm{Spec}(R)$ where $R = k[x,y]/(y^2 - x^3 - x^2)$ and k is a field. There is an isomorphism of k-algebras

$$R \xrightarrow{\sim} k[t^2, t^3 - t] \subset k[t], \quad x \mapsto t^2 - 1, \quad y \mapsto t(t^2 - 1).$$

Since t^2 comes from R, t is integral over R, so that $k[t]$ is an integral (even finite) R-algebra. Moreover, the map $R \hookrightarrow k[t]$ becomes an isomorphism after inverting x, so that R and $k[t]$ have the same field of fractions $k(t)$. As $k[t]$ is integrally closed in $k(t)$, we deduce that $k[t]$ is precisely the integral closure of R in $k(t)$. Hence, the normalization of X is $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$.

For a non-affine example, consider the projective closure $\bar{X} = \mathrm{Proj}(k[x, y, z]/(y^2z - x^3 - x^2z))$ of X in $\mathbb{P}^2_k$. The complement of the node (the locus where $x = 1$) is isomorphic to $\mathrm{D}(y^2 - 1) \subset \mathrm{Spec}(k[y])$, which is already normal. Hence, the normalization of $\bar{X}$ is $\mathbb{P}^1_k$.

Proposition 8.87 (Characterization of integrally closed noetherian domains)**.** *A noetherian integral domain R is integrally closed if and only if the following conditions hold:*

(i) *For every prime ideal $\mathfrak{p} \subset R$ of codimension 1, $R_\mathfrak{p}$ is a discrete valuation ring (or equivalently, by Example 8.69, a regular local ring).*
(ii) *R is the intersection of its localizations $R_\mathfrak{p} \subset \mathrm{Frac}(R)$ at the prime ideals $\mathfrak{p}$ of codimension 1.*

Corollary 8.88. *Let X be a locally noetherian normal scheme.*

(i) *For every point $x \in |X|$ of codimension ≤ 1, the local ring $\mathcal{O}_{X,x}$ is regular.*
(ii) (Hartog's theorem) *Let $U \subset X$ be an open subscheme such that every point in the complement has codimension ≥ 2. Then the map $\mathcal{O}(X) \to \mathcal{O}(U)$ is bijective, i.e., every function on U extends uniquely to X.*

Example 8.89 (Dedekind schemes)**.** Let X be a locally noetherian scheme of Krull dimension ≤ 1. By Corollary 8.88(i), X is normal if and only if it is regular. Such a scheme X is called a *Dedekind scheme*. An affine scheme $\mathrm{Spec}(R)$ is a Dedekind scheme if and only if R is a finite product of Dedekind domains.

If X is any locally noetherian integral scheme of Krull dimension ≤ 1, the Krull–Akizuki theorem implies that the normalization X^ν is again locally noetherian, and is therefore a Dedekind scheme.

Remark 8.90 (Serre's conditions)**.** Let X be a locally noetherian scheme. We can consider the following conditions for all $k \in \mathbb{N}$:

(R_k) For every point $x \in |X|$ of codimension $\leq k$, $\mathcal{O}_{X,x}$ is regular.
(S_k) For every point $x \in |X|$, $\mathrm{depth}(\mathcal{O}_{X,x}) \geq \min\{k, \mathrm{codim}(x, X)\}$

Note that these are local properties by definition, and that $(\mathrm{R}_k) \Rightarrow (\mathrm{R}_{k+1})$ and $(\mathrm{S}_k) \Rightarrow (\mathrm{S}_{k+1})$. These conditions can be used to give a uniform treatments of various properties of locally noetherian schemes:

(i) X is reduced if and only if it satisfies (R_0) and (S_1).
(ii) X is normal if and only if it satisfies (R_1) and (S_2).
(iii) X is regular if and only if it satisfies (R_k) for all k.
(iv) X is Cohen–Macauley if and only if it satisfies (S_k) for all k.

9. Smoothness

9.1. **Smooth and étale morphisms.** Smooth morphisms in algebraic geometry are the analogue of *submersions* in differential geometry, i.e., morphisms along which tangent vectors (and more general tangential data) can be lifted. In particular, there is no relation to the usage of "smooth" to mean "C^∞" in analysis (all maps in algebraic geometry are "smooth" in that sense, since they are polynomial maps). Étale morphisms are smooth morphisms of relative dimension zero, and they are analogous to *local diffeomorphisms* in differential geometry (indeed, submersions of relative dimension zero are local diffeomorphisms by the inverse function theorem).

We begin with some recollections from differential geometry. If $f_1, \dots, f_m$ are C^∞-functions on $\mathbb{R}^n$, the implicit function theorem gives a sufficient condition under which the vanishing locus

$$X = \mathrm{V}(f_1, \dots, f_m) \subset \mathbb{R}^n$$

acquires a structure of C^∞-manifold: it suffices that, for every point $p \in X$, the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right) \in \mathrm{Mat}_{m\times n}(\mathbb{R})$$

have rank m. In that case, the m rows of this matrix form a basis of the normal space to X at p, whose orthogonal complement in $\mathbb{R}^n$ is the tangent space to X at p.

Remark 9.1. Like in algebraic geometry, the vanishing locus $X = \mathrm{V}(f_1, \dots, f_m) \subset \mathbb{R}^n$ is a priori a (possibly singular) geometric object called a "C^∞-scheme". The Jacobian criterion above is then more precisely a sufficient condition for the map of C^∞-schemes $X \to *$ to be a submersion.

Remark 9.2. A peculiar feature of real fields k is that the standard scalar product on k^n remains non-degenerate on any subspace of k^n (this is not the case over other fields, not even over $\mathbb{C}$!). This implies that every subspace of k^n is canonically isomorphic to its dual space, which can cause some confusion. Generalizing the above discussion away from the real numbers reveals that the row space of the Jacobian matrix is more canonically the *conormal* space, i.e., the dual of the normal space, whose elements are the linear forms that vanish on tangent vectors.

We can copy this Jacobian criterion to define smooth morphisms in algebraic geometry:

Definition 9.3 (Smooth and étale morphisms)**.** Let $f\colon Y \to X$ be a morphism of schemes. We say that f is *smooth* if, for every point $y \in |Y|$, there are affine open subschemes $V \subset Y$ and $U \subset X$ with $y \in |V|$ and $f(V) \subset U$ such that $f|_V$ factors as

$$V \xrightarrow{\sim} \mathrm{V}(f_1, \dots, f_m) \subset \mathbb{A}^n_U \to U,$$

for some $n, m \in \mathbb{N}$ and some functions $f_1, \dots, f_m \in \mathcal{O}(\mathbb{A}^n_U) = \mathcal{O}(U)[x_1, \dots, x_n]$ such that, for every $v \in |V|$, the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}(v)\right)_{i,j} \in \mathrm{Mat}_{m\times n}(\kappa(v))$$

has rank m. We say that f is *étale* if the same statement holds with $m = n$. Note that smooth and étale morphisms are locally of finite presentation, by Example 8.10.

Definition 9.3, while fairly concrete, is difficult to work with. For example, it is not at all obvious that smoothness is a local property, and it is not easy to show from this definition that a morphism is *not* smooth. In this chapter, we will establish several characterizations of smoothness and étaleness, which address these issues. We record and sloganize the most important ones in the following theorems (flatness was defined in Example 8.11; the cotangent and conormal modules are defined in §**??** and formal smoothness is defined in §**??**):

Theorem 9.4 (Characterizations of smoothness)**.** *Let $f\colon Y \to X$ be a morphism of schemes locally of finite presentation. The following assertions are equivalent:*

(i) (Jacobian criterion) *f is smooth.*
(ii) (Infinitesimal lifting property) *f is formally smooth.*
(iii) (Cotangent complex is a vector bundle) *The contangent module $\Omega_f \in \mathrm{Mod}_Y$ is a vector bundle and the conormal module $\mathcal{N}_f \in \mathrm{Mod}_Y$ is zero.*
(iv) (Flat and cotangent module is a vector bundle) *f is flat and the cotangent module $\Omega_f \in \mathrm{Mod}_Y$ is a vector bundle of rank $y \mapsto \dim(Y_{f(y)})$.*
(v) (Flat and geometric fibers are regular) *f is flat and for every algebraically closed field k and every map $x\colon \mathrm{Spec}(k) \to X$, the scheme Y_x is regular.*

Theorem 9.5 (Characterizations of étaleness)**.** *Let $f\colon Y \to X$ be a morphism of schemes locally of finite presentation. The following assertions are equivalent:*

(i) (Jacobian criterion) *f is étale.*
(ii) (Unique infinitesimal lifting property) *f is formally étale.*
(iii) (Cotangent complex vanishes) *The cotangent module $\Omega_f \in \mathrm{Mod}_Y$ and the conormal module $\mathcal{N}_f \in \mathrm{Mod}_Y$ are both zero.*
(iv) (Flat and cotangent module vanishes) *f is flat and the cotangent module $\Omega_f \in \mathrm{Mod}_Y$ is zero.*
(v) (Flat and geometric fibers are sets) *f is flat and for every algebraically closed field k and every map $x\colon \mathrm{Spec}(k) \to X$, the k-scheme Y_x is a coproduct of copies of* $\mathrm{Spec}(k)$.

Corollary 9.6. *"Smooth" and "étale" are local properties of morphisms of schemes that are stable under base change and composition.*

Example 9.7. Let k be a ring and consider the affine curve $C = \mathrm{Spec}(k[x,y]/(y^2 - x^3 + x))$ from §1.1. By definition, we have $C \simeq \mathrm{V}(y^2 - x^3 + x) \subset \mathbb{A}^2_k$. The corresponding Jacobian matrix is the 1×2-matrix $(-3x^2 + 1, 2y)$ over $k[x,y]$. A point $v \in |C|$ corresponds to a pair $(a,b) \in \kappa(v)^2$ with $b^2 = a^3 - a$, and evaluating the matrix at v yields the matrix $(-3a^2 + 1, 2b)$ over $\kappa(v)$. This matrix only fails to have rank 1 if $\mathrm{char}(\kappa(v)) = 2$ and $(a,b) = (1,0)$. Thus, if k is a $\mathbb{Z}[\frac{1}{2}]$-algebra, then the map $C \to \mathrm{Spec}(k)$ is smooth by Definition 9.3.

If on the other hand 2 is not a unit in k, then $C \to \mathrm{Spec}(k)$ is *not* smooth: one can see this via Theorem 9.4(v), by computing that the Zariski cotangent space at the k-point $(1,0)$ has dimension 2 when k is a field of characteristic 2, so that the base change of C to any such field is not regular.

Example 9.8 (Finite separable field extensions are étale)**.** Let k be a field and let $k \subset k'$ be a finite separable field extension. By the primitive element theorem, there is an $a \in k'$ such that $k' = k[a]$. The choice of a is equivalent to a choice of closed immersion $\mathrm{Spec}(k') \hookrightarrow \mathbb{A}^1_k$ over $\mathrm{Spec}(k)$. Let $m \in k[t]$ be the minimal polynomial of a. As a is separable over k, a is not a multiple root of m, so that $\frac{\partial m}{\partial t}(a) \neq 0$ in k'. By Definition 9.3, this shows that $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is étale.

Conversely, if $k \subset k'$ is a finite field extension such that $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is smooth, then k' is a separable extension of k. We can see this via Theorem 9.4(v) and the fact that if k' is not separable over k and $\bar{k}$ is an algebraically closed extension of k, then $k' \otimes_k \bar{k}$ is not reduced and hence not regular.

9.2. **Cotangent and conormal modules.**

Definition 9.9 (Derivation)**.** Let $R \to S$ be a ring map and let M be an S-module. An *R-derivation* of S into M is a map (of sets) $\delta\colon S \to M$ such that:

(i) δ is R-linear;
(ii) δ satisfies the *Leibniz rule*: for every $a, b \in S$, $\delta(ab) = a\delta(b) + b\delta(a)$.

We denote by $\mathrm{Der}_R(S, M)$ the set of R-derivations of S into M, which is an S-module under pointwise addition and scalar multiplication.

Remark 9.10. An R-derivation $\delta\colon S \to M$ is equivalently a $\mathbb{Z}$-derivation such that $\delta|_R = 0$.

Remark 9.11 (Split square-zero extensions)**.** The following alternative description of derivations is often useful. If M is an S-module, the *split square-zero extension* of S by M is the ring whose underlying abelian group is $S \oplus M$ with multiplication given by

$$(a, m)(b, n) = (ab, bm + an).$$

Note that both canonical maps $S \hookrightarrow S \oplus M \twoheadrightarrow S$ are ring maps, and that $M \hookrightarrow S \oplus M$ is a square-zero ideal. There is then a bijection

$$\mathrm{Der}_R(S, M) \xrightarrow{\sim} \{R\text{-algebra sections of } S \oplus M \twoheadrightarrow S\}, \quad \delta \mapsto (\mathrm{id}_S, \delta).$$

Definition 9.12 (Cotangent module, differential forms)**.** Let $R \to S$ be a ring map. We denote by $\mathrm{d}\colon S \to \Omega_{S/R}$ the *universal* R-derivation of S, i.e., the R-derivation such that, for every R-derivation $\delta\colon S \to M$, there is a unique S-linear map $f\colon \Omega_{S/R} \to M$ with $f \circ \mathrm{d} = \delta$:

$$\begin{array}{ccc} S & \xrightarrow{\mathrm{d}} & \Omega_{S/R} \\ & {\scriptstyle\forall\delta}\searrow & \Big\downarrow {\scriptstyle\exists!} \\ & & M. \end{array}$$

The S-module $\Omega_{S/R}$ is called the *cotangent module* of S over R, and its elements are called the *differential forms* or simply the *differentials* of S over R.

Remark 9.13. The functor $\mathrm{Der}_R(S, -)\colon \mathrm{Mod}_S \to \mathrm{Mod}_S$ has a left adjoint given by $M \mapsto \Omega_{S/R} \otimes_S M$.

Proposition 9.14 (Presentation of the cotangent module)**.** *Let R be a ring and let S be an R-algebra with a presentation $S = R[x_i \,|\, i \in I]/(f_j \,|\, j \in J)$. Then the S-module $\Omega_{S/R}$ is generated by the differentials $\mathrm{d}x_i$ for all $i \in I$ with relations given by $\sum_{i \in I} \frac{\partial f_j}{\partial x_i} \mathrm{d}x_i$ for all $j \in J$.*

Corollary 9.15 (Finiteness of the cotangent module)**.** *If $R \to S$ is a morphism of finite presentation (resp. of finite type), then the S-module $\Omega_{S/R}$ is of finite presentation (resp. of finite type).*

Remark 9.16 (The de Rham complex)**.** As in differential geometry, we define *differential n-forms* for $n \in \mathbb{N}$ by taking exterior powers of the cotangent module: $\Omega^n_{S/R} = \bigwedge^n_S \Omega_{S/R}$. The universal derivation d then extends to a cochain complex of S-modules

$$\Omega^0_{S/R} = S \xrightarrow{\mathrm{d}} \Omega^1_{S/R} = \Omega_{S/R} \xrightarrow{\mathrm{d}} \Omega^2_{S/R} \xrightarrow{\mathrm{d}} \Omega^3_{S/R} \xrightarrow{\mathrm{d}} \cdots, \quad \mathrm{d}(g\, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n) = \mathrm{d}g \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n,$$

which is a commutative differential graded S-algebra under the exterior product. This complex is called the *de Rham complex* of S over R, and its cohomology is called the *de Rham cohomology* of S over R.

We now discuss the functoriality of the cotangent module. Consider a commutative square of rings

$$\begin{array}{ccc} R & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ R' & \longrightarrow & S'. \end{array}$$

The map $S \to S' \xrightarrow{\mathrm{d}} \Omega_{S'/R'}$ is then an R-derivation of S, which induces an S-linear map $\Omega_{S/R} \to \Omega_{S'/R'}$, or equivalently an S'-linear map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R'}$. This construction is easily seen to be compatible with vertical composition of squares. To make this precise, consider the following category CAlgMod:

- The objects of CAlgMod are pairs (R, M) where R is a ring and M is an R-module.
- A morphism $(R, M) \to (S, N)$ in CAlgMod is a pair (φ, f) where $\varphi\colon R \to S$ is a ring map and f is an R-linear map $M \to N$ (or equivalently an S-linear map $M \otimes_R S \to N$).
- Composition is defined componentwise.

There is a forgetful functor CAlgMod $\to$ CAlg sending (R, M) to R (also known as the *cocartesian fibration* classified by the functor $\mathrm{Mod}^*\colon \mathrm{CAlg} \to \mathrm{Cat}$ of Example 4.3(i)). We can then view the assignment $(R \to S) \mapsto (S, \Omega_{S/R})$ as a functor making the following triangle commute:

$$\begin{array}{ccc} & & \mathrm{CAlgMod} \\ & \overset{\Omega}{\nearrow} & \downarrow \\ \mathrm{Ar}(\mathrm{CAlg}) & \xrightarrow[\text{target}]{} & \mathrm{CAlg}. \end{array}$$

Proposition 9.17 (Properties of the cotangent module)**.** *Let $R \to S$ be a ring map*

(i) (Base change) *For any base change $R' \to S'$ of $R \to S$, the S'-linear map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R'}$ is an isomorphism.*

(ii) (Localization) *For any localization $S \to S'$, the S'-linear map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$ is an isomorphism.*

(iii) (Quotients) *For any surjection $S \twoheadrightarrow S'$ with kernel I, the following sequence of S'-modules is exact:*

$$I/I^2 \xrightarrow{\mathrm{d}} \Omega_{S/R} \otimes_S S' \to \Omega_{S'/R} \to 0.$$

Corollary 9.18. *If $R \to S$ is an epimorphism in* CAlg*, for example a surjection or a localization, then $\Omega_{S/R} = 0$.*

Proposition 9.19 (The fundamental exact sequence of cotangent modules)**.** *Let $R \to S \to T$ be ring maps. Then the induced sequence of T-modules*

$$\Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to \Omega_{T/S} \to 0$$

is exact.

Example 9.20. Let k be a ring and consider the k-algebra maps

$$k \xrightarrow{f} k[x,y]/(y^2 - x^3) \overset{g}{\hookrightarrow} k[t], \quad g(x) = t^2, \quad g(y) = t^3.$$

By Proposition 9.14, we have

$$\Omega_{gf} = k[t]\mathrm{d}t \quad \text{and} \quad g^*\Omega_f = (k[t]\mathrm{d}x \oplus k[t]\mathrm{d}y)/(2t^3\mathrm{d}y - 3t^4\mathrm{d}x),$$

and the $k[t]$-linear map $g^*\Omega_f \to \Omega_{gf}$ sends $\mathrm{d}x$ to $2t\mathrm{d}t$ and $\mathrm{d}y$ to $3t^2\mathrm{d}t$. By Proposition 9.19 we obtain the following presentation of Ω_g:

$$\Omega_g = \mathrm{coker}(g^*\Omega_f \to \Omega_{gf}) = (k[t]\mathrm{d}t)/(2t\mathrm{d}t, 3t^2\mathrm{d}t).$$

Note that the map $g^*\Omega_f \to \Omega_{gf}$ also has a nontrivial kernel (if $k \neq 0$), as it identifies $3t\mathrm{d}x$ and $2\mathrm{d}y$.

As Example 9.20 shows, the fundamental exact sequence of Proposition 9.19 is not always a short exact sequence. Our next goal is to extend this sequence to the left, generalizing Proposition 9.17(iii).

Construction 9.21. Let $R \to S$ be a ring map and let $\pi\colon P \twoheadrightarrow S$ be a surjection where P is a polynomial R-algebra. Let $I \subset P$ be the kernel of π. By the Leibniz rule, the universal derivation $\mathrm{d}\colon P \to \Omega_{P/R}$ induces an S-linear map

$$\mathrm{d}_\pi\colon I/I^2 \to \Omega_{P/R} \otimes_P S,$$

whose cokernel is $\Omega_{S/R}$ by Proposition 9.17(iii). We denote the kernel of d_π by $\mathcal{N}_{S/R}(\pi)$.

Lemma 9.22. *Given two polynomial R-algebras P and Q and a commuting triangle*

$$\begin{array}{ccc} P & & \\ & \overset{\pi}{\searrow\!\!\!\searrow} & \\ \downarrow & & S, \\ & \underset{\rho}{\nearrow\!\!\!\nearrow} & \\ Q & & \end{array}$$

the induced map $\mathcal{N}_{S/R}(\pi) \to \mathcal{N}_{S/R}(\rho)$ is an isomorphism.

Thus, the S-module $\mathcal{N}_{S/R}(\pi)$ is canonically independent of π. For definiteness, we can define the conormal module using the canonical choice of π:

Definition 9.23 (Conormal module)**.** Let $R \to S$ be a ring map and let $\pi\colon R[x_s \mid s \in S] \twoheadrightarrow S$ be the R-algebra map with $\pi(x_s) = s$. The S-module $\mathcal{N}_{S/R}(\pi)$ is called the *conormal module* of S over R and is denoted by $\mathcal{N}_{S/R}$.

Example 9.24 (Conormal modules of polynomial and quotient algebras)**.** We can compute the conormal module $\mathcal{N}_{S/R}$ using any choice of $\pi\colon P \twoheadrightarrow S$, by Lemma 9.22. For example:

(i) If P is any polynomial R-algebra, then $\mathcal{N}_{P/R} = 0$ (choose $\pi = \mathrm{id}_P$).
(ii) If $R \twoheadrightarrow S$ is a surjection with kernel I, then $\mathcal{N}_{S/R} = I/I^2$ (choose $P = R$).

The conormal module has the same functoriality in $R \to S$ as the cotangent module:

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R' & \longrightarrow & S' \end{array} \quad\Longrightarrow\quad \begin{array}{ccccc} R & \longrightarrow & P & \overset{\pi}{\twoheadrightarrow} & S \\ \downarrow & & \downarrow & & \downarrow \\ R' & \longrightarrow & P' & \overset{\pi'}{\twoheadrightarrow} & S' \end{array} \quad\Longrightarrow\quad \begin{array}{ccccc} \mathcal{N}_{S/R} & \longrightarrow & I/I^2 & \xrightarrow{\mathrm{d}\pi} & \Omega_{P/R} \otimes_P S \\ \vdots & & \downarrow & & \downarrow \\ \mathcal{N}_{S'/R'} & \longrightarrow & I'/I'^2 & \xrightarrow{\mathrm{d}\pi'} & \Omega_{P'/R'} \otimes_{P'} S', \end{array}$$

where $P = R[x_s \,|\, s \in S]$, $P' = R'[x_t \,|\, t \in S']$, $I = \ker(\pi)$, $I' = \ker(\pi')$, and $P \to P'$ is induced by $S \to S'$.

Proposition 9.25 (Properties of the conormal module)**.** *Let $R \to S$ be a ring map.*

(i) (Base change) *For any base change $R' \to S'$ of $R \to S$, the S'-linear map $\mathcal{N}_{S/R} \otimes_S S' \to \mathcal{N}_{S'/R'}$ fits in an exact sequence*
$$\mathrm{Tor}_2^S(\Omega_{S/R}, S') \to \mathcal{N}_{S/R} \otimes_S S' \to \mathcal{N}_{S'/R'} \to \mathrm{Tor}_1^S(\Omega_{S/R}, S') \to 0.$$
In particular, if either $\Omega_{S/R}$ or S' is flat over S, then $\mathcal{N}_{S/R} \otimes_S S' \xrightarrow{\sim} \mathcal{N}_{S'/R'}$.
(ii) (Localization) *For any localization $S \to S'$, the S'-linear map $\mathcal{N}_{S/R} \otimes_S S' \to \mathcal{N}_{S'/R}$ is an isomorphism.*

Corollary 9.26. *If $R \to S$ is a flat epimorphism in* CAlg*, for example a localization, then $\mathcal{N}_{S/R} = 0$.*

Theorem 9.27 (The fundamental exact sequence of cotangent and conormal modules)**.** *Let $R \to S \to T$ be ring maps. Then there is an exact sequence of T-modules*
$$? \to \mathcal{N}_{T/R} \to \mathcal{N}_{T/S} \to \Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to \Omega_{T/S} \to 0,$$
where ? fits in an exact sequence
$$\mathrm{Tor}_2^S(\Omega_{S/R}, T) \to \mathcal{N}_{S/R} \otimes_S T \to \,?\, \to \mathrm{Tor}_1^S(\Omega_{S/R}, T) \to 0.$$
In particular, if $\mathrm{Tor}_1^S(\Omega_{S/R}, T) = 0$ (e.g., if either $\Omega_{S/R}$ or T is flat over S), then there is an exact sequence
$$\mathcal{N}_{S/R} \otimes_S T \to \mathcal{N}_{T/R} \to \mathcal{N}_{T/S} \to \Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to \Omega_{T/S} \to 0.$$

Remark 9.28 (The cotangent complex)**.** The exact sequence of Theorem 9.27 can be extended further to a long exact sequence, which (like all long exact sequences) comes from a *fiber sequence* in a higher category. Namely, if we repeat the definition of the cotangent module $\Omega_{S/R}$ but replace the category Mod_S with its derived category $\mathrm{D}(\mathrm{Mod}_S)$, we obtain the definition of the *cotangent complex* $\mathrm{L}_{S/R} \in \mathrm{D}(\mathrm{Mod}_S)$, such that
$$\mathrm{H}_0(\mathrm{L}_{S/R}) = \Omega_{S/R}, \quad \mathrm{H}_1(\mathrm{L}_{S/R}) = \mathcal{N}_{S/R}, \quad \mathrm{H}_{<0}(\mathrm{L}_{S/R}) = 0.$$
The proof of Proposition 9.19 carries over to this setting: for every ring maps $R \to S \to T$, we obtain a fiber sequence
$$\mathrm{L}_{S/R} \otimes_S T \to \mathrm{L}_{T/R} \to \mathrm{L}_{T/S}$$
in $\mathrm{D}(\mathrm{Mod}_T)$, hence a long exact sequence of homology modules. This explains the mysterious module in Theorem 9.27: $? = \mathrm{H}_1(\mathrm{L}_{S/R} \otimes_S T)$.