# SOSE 2026 SEMINAR: MOTIVIC COHOMOLOGY OF SCHEMES

The goal of this seminar is to understand the motivic filtration of the $\mathbb{A}^1$-invariant algebraic K-theory of arbitrary schemes and the resulting theory of $\mathbb{A}^1$-invariant motivic cohomology, recently developed in this generality by Bachmann, Elmanto, and Morrow [BEM]. The construction uses many aspects of the formalism of motivic spectra, including the slice filtration, the six-functor formalism, and framed correspondences, which we will review. We will then discuss the comparison with cycle-theoretic constructions in special cases (for smooth schemes over Dedekind rings) and with syntomic cohomology (the Beilinson–Lichtenbaum conjecture), and compute motivic cohomology in low weights.

(1) **Motivic spectra, six functors, and cdh descent (14.04)** Review the construction of the Morel–Voevodsky category of motivic spectra $\mathrm{SH}(X)$ and of the motivic spectrum KGL representing $\mathbb{A}^1$-invariant K-theory KH. Define the cdh topology on schemes and use the six-functor formalism on SH to show that SH and KH are cdh sheaves.

(2) **The slice filtration, the main theorem, and Voevodsky's slice conjectures (21.04)** Define the slice filtration on SH and the three cohomology theories $\mathbb{Z}(\star)^{\mathbb{A}}$, $\mathbb{Z}(\star)^{\mathbb{A},\mathrm{cdh}}$, and $\mathbb{Z}(\star)^{\mathrm{cdh}}$ on schemes [BEM, §3.3, §4.2, §7.2]. Record the key properties [BEM, Propositions 3.44 and 3.45]. State the main theorem [BEM, Theorem 9.1], and deduce its immediate consequences [BEM, Corollary 9.2, Theorem 9.3, Corollary 9.5, Corollary 9.10].

(3) **Framed correspondences (28.04)** Sketch the definition of the category of framed correspondences [EHK+21, §3.2], and state the recognition principle over perfect fields [EHK+21, Theorem 3.5.14] and the reconstruction theorem $\mathrm{SH}(X) \simeq \mathrm{SH}^{\mathrm{fr}}(X)$ [Hoy21, Theorem 18]. Use the latter to define the absolute motivic spectrum $X \mapsto \mathcal{V}_X$ [BEM, §4.1.3] such that $\mathcal{V}_k \simeq \mathrm{kgl}_k$ for any field $k$ and $\mathcal{V}_X[\beta^{-1}] \simeq \mathrm{KGL}_X$ for any $X$ [HJN+25, §5].

(4) **The zeroth slice of the motivic sphere (05.05)** Define Levine's homotopy coniveau tower [Lev08, §2.1] and explain its relation to Voevodsky's slice filtration [Lev08, Theorem 9.0.3]. Use this to compute the slices of KGL over a field in terms of Bloch's cycle complex [Lev08, Theorem 6.4.2]. Then state Levine's result that $s^0(1) \simeq s^0(\mathrm{KGL})$, and sketch the proof by Bachmann–Elmanto using framed correspondences [BE21, §4].

(5) **Rational splitting (12.05)** Define the Adams operation $\psi^k$ on $\mathrm{KGL}[\frac{1}{k}]$ and sketch the proof of Riou's decomposition theorem for $\mathrm{KGL} \otimes \mathbb{Q}$ [BEM, Theorem 4.48]. Deduce the stability under base change and multiplicative splitting of the slice filtration on $\mathrm{KGL} \otimes \mathbb{Q}$ [BEM, Theorem 4.47], and list some consequences [BEM, Corollary 4.52].

(6) **Low weights (19.05)** Explain the computation of the first two stages of the unstable slice filtration $\mathrm{Fil}^1_{\text{u-slice}}\mathrm{K}$ and $\mathrm{Fil}^2_{\text{u-slice}}\mathrm{K}$ over regular schemes [BEM, Lemma 4.32]. Deduce that $\mathbb{Z}(\star)^{\mathrm{cdh}}$ is $\mathbb{Z}$-linear and define the first Chern class [BEM, Construction 4.35]. Rationally, state [BEM, Corollary 4.60(1)]. Finally, taking for granted that the first Chern class is an isomorphism over Dedekind domains [BEM, Corollary 6.26] and the equivalence $\mathbb{Z}(j)^{\mathrm{cdh}} \simeq \mathbb{Z}(j)^{\mathbb{A}}$ for $j \leq 1$, show that $\mathrm{Fil}^1_{\mathrm{slice}}\mathrm{KH} = L_{\mathrm{cdh}}\mathrm{fib}(\mathrm{rk}\colon \mathrm{K} \to \mathbb{Z}[0])$ and $\mathrm{Fil}^2_{\mathrm{slice}}\mathrm{KH} = L_{\mathrm{cdh}}\mathrm{fib}(\det\colon \mathrm{K} \to \mathrm{Pic}^{\dagger})$ over arbitrary schemes [BEM, Corollary 9.12(1)].

(7) **Syntomic cohomology (02.06)** Review the definitions of the syntomic cohomology of $p$-complete rings [BEM, §5.1] and of its globalization $\mathbb{Z}_p(\star)^{\mathrm{syn}}$ to schemes via the syntomic-to-étale comparison map $\gamma^{\text{ét}}_{\mathrm{syn}}\{\star\}$ [BEM, §5.2]. Sketch the proof that $\mathbb{Z}_p(\star)^{\mathrm{syn}}$ is an fpqc sheaf and left Kan extended from smooth $\mathbb{Z}$-algebras [BEM, Proposition 5.20] and the computation of its $\mathbb{A}^1$-localization and h-sheafification [BEM, Theorem 5.33]. Define the first syntomic Chern class and state the computation of the Milnor range [BEM, Theorem 5.42].

(8) **Beilinson–Lichtenbaum cohomology (09.06)** Define the cohomology theory $\mathbb{Z}_p(\star)^{\mathrm{BL}}$ and prove that it is $\mathbb{A}^1$-invariant and satisfies the $\mathbb{P}^1$-bundle formula for smooth schemes over Dedekind domains [BEM, Theorem 5.47] (first deal with the case of fields, and then use the localization sequence [BEM, Theorem 5.58] to reduce to that case).

(9) **Axiomatic Beilinson–Lichtenbaum isomorphism (16.06)** Introduce the "Beilinson–Lichtenbaum" subcategory $\mathfrak{BL}_p \subset \mathrm{Sch}^{\mathrm{qcqs}}$. Review the classical result that smooth schemes over a field belong to $\mathfrak{BL}_p$ [BEM, Theorem 6.3]. Then sketch the proof that $\mathbb{Z}_p(\star)^{\mathbb{A}} \simeq \mathbb{Z}_p(\star)^{\mathrm{BL}}$ on $\mathfrak{BL}_p$ [BEM, Theorem 6.14].

(10) **Beilinson–Lichtenbaum over Dedekind domains (23.06)** Define the motivic spectrum $H\mathbb{Z}_p^{\mathrm{BL}}$ over Dedekind schemes. Recall the rigidity property of syntomic cohomology, which implies that $H\mathbb{Z}_p^{\mathrm{BL}}$ is stable under base change [BEM, Theorem 6.17]. Show that $s^0(1)_p^\wedge \simeq H\mathbb{Z}_p^{\mathrm{BL}}$ [BEM, Proposition 6.18] and deduce [BEM, Proposition 6.20] (which we already know over fields); time permitting, deduce also the generalization to arbitrary schemes [BEM, Corollary 9.7], assuming [BEM, Theorem 9.3]. Then sketch the proof that smooth schemes over Dedekind domains belong to $\mathfrak{BL}_p$ [BEM, Theorem 6.21] and hence satisfy the Beilinson–Lichtenbaum equivalence [BEM, Corollary 6.24].

(11) **Cdh-motivic cohomology (30.06)** Briefly review some of the key properties of $\mathbb{Z}(\star)^{\mathrm{cdh}}$ from [BEM, Theorems 7.12 and 7.16], which follow from previously established results for smooth $\mathbb{Z}$-schemes. Explain in particular the concrete descriptions of $\mathbb{Z}(\star)^{\mathrm{cdh}}/p^r$ on $\mathbb{F}_p$-schemes [BEM, Corollary 7.19] and on $\mathbb{Z}[\frac{1}{p}]$-schemes [BEM, Corollary 7.20]. Then prove the key result that $\mathbb{Z}(\star)^{\mathrm{cdh}}$ satisfies Milnor excision [BEM, Theorem 7.21].

(12) **$\mathbb{A}^1$-invariance and projective bundle formula (07.07)** Introduce the "key hypothesis" [BEM, §8.1], and review its status [BEM, Example 8.6, Proposition 8.7]. Show that the key hypothesis implies that $\mathbb{Z}(\star)^{\mathrm{cdh}}$ is $\mathbb{A}^1$-invariant [BEM, Theorem 8.14] and satisfies the projective bundle formula [BEM, Theorem 8.20]. The two proofs follow a similar strategy, so one may focus for example on the proof of $\mathbb{A}^1$-invariance.

(13) **Kato motivic cohomology and proof of the main theorem (14.07)** Give an overview of the proof of the main theorem [BEM, §9.3].

## References


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