SOSE 2026 SEMINAR: COHOMOLOGY OF SHEAVES AND SCHEMES

In this seminar, we will introduce the cohomology of sheaves on topological spaces and schemes following Chapter III of Hartshorne [Har97]. Beyond the definitions and basic properties of sheaf cohomology, we will compute the cohomology of affine schemes and projective spaces, and we will discuss the Serre duality theorem for projective schemes and its connection to the residue theorem in complex analysis. The first five talks are about sheaves on topological spaces and do not require any background in algebraic geometry. The rest assumes some basic knowledge of schemes (e.g., from Algebraic Geometry I or Chapter II of Hartshorne).

Unless otherwise specified, all references to [Har97] below refer to Chapter III.

(1) **Introduction (15.04)**

Part I: Sheaf cohomology

(2) **Derived functors I: Construction (22.04)** Recall the notion of abelian category and left/right exact functors. Give some examples, including the category of quasi-coherent modules over a scheme. For a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories where $\mathcal{A}$ has enough injectives, sketch the construction of the right derived functors $\mathrm{R}^i F\colon \mathcal{A} \to \mathcal{B}$ and the proof of the properties listed in [Har97, Theorem 1.1A], and explain the corollary [Har97, Proposition 1.2A]. Proofs may be found in [HS97, IV, §1–6], [Wei94, §2.4], or [Hoy, §4.3.1]. (Note: δ-functors are introduced in the next talk.)

(3) **Derived functors II: Universal property (29.04)** Introduce the notion of δ-*functor* between abelian categories, and of *universal* δ-functor. Prove Grothendieck's theorem that effaceable/erasable δ-functors are universal [Har97, Theorem 1.3A] (for a proof, modify the proof of [Wei94, Theorem 2.4.6] as suggested in [Wei94, Exercise 2.4.5]). Deduce [Har97, Corollary 1.4].

(4) **Cohomology of sheaves (06.05)** Prove that the category $\mathrm{Mod}(X)$ of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ has enough injectives [Har97, Proposition 2.2]. Define *sheaf cohomology* of a topological space X as the right derived functors of $\Gamma(X, -)\colon \mathrm{Ab}(X) \to \mathrm{Ab}$. Show that injective $\mathcal{O}_X$-modules are flasque [Har97, Lemma 2.4] and that flasque sheaves are acyclic [Har97, Proposition 2.5]. Deduce [Har97, Proposition 2.6].

(5) **Čech cohomology (13.05)** Define the *Čech cohomology* groups $\check{\mathrm{H}}^*(\mathfrak{U}, \mathcal{F})$ for an open covering $\mathfrak{U}$ of X and $\mathcal{F} \in \mathrm{Ab}(X)$, as well as the groups $\check{\mathrm{H}}^*(X, \mathcal{F}) = \mathrm{colim}_{\mathfrak{U}} \check{\mathrm{H}}^*(\mathfrak{U}, \mathcal{F})$. Prove the vanishing of Čech cohomology for flasque sheaves [Har97, Proposition 4.3]. Construct the comparison map $\check{\mathrm{H}}^n(X, \mathcal{F}) \to \mathrm{H}^n(X, \mathcal{F})$ [Har97, Lemma 4.4]. Show that it is an isomorphism for $n = 0$ [Har97, Lemma 4.1], and for $n = 1$ [Stacks, Tags 03AJ and 0A6G]. Mention (without proof) that it is an isomorphism for all n if X is a paracompact Hausdorff space.

(6) **Grothendieck vanishing (27.05)** Prove Grothendieck's vanishing theorem [Har97, Theorem 2.7]: if X is a noetherian topological space of Krull dimension n, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i > n$ and $\mathcal{F} \in \mathrm{Ab}(X)$.

Part II: Cohomology of schemes

(7) **Cohomology of affine schemes (03.06)** Prove that $\mathrm{H}^n(X, \mathcal{F}) = 0$ for X a noetherian affine scheme, $\mathcal{F}$ a quasi-coherent sheaf, and $n > 0$ [Har97, Theorem 3.5] (assuming Krull's theorem [Har97, Proposition 3.1A]). Explain the corollary [Har97, Corollary 3.6].

(8) **Serre's affineness criterion and cohomology of separated schemes (10.06)** Prove Serre's cohomological criterion for affineness [Har97, Theorem 3.7], which is a converse of the vanishing result of the previous talk (in particular, explain [Har97, Ch. II, Exercise 2.17(b)]). Then, show that the Čech cohomology of affine open coverings computes sheaf cohomology for noetherian separated schemes [Har97, Theorem 4.5], which combines results from talks (4), (5), and (7).

(9) **Cohomology of projective spaces (17.06)** Prove [Har97, Theorem 5.1], computing in particular the cohomology groups $\mathrm{H}^*(\mathbb{P}^n_A, \mathcal{O}(d))$.

(10) **Serre vanishing (24.06)** Prove [Har97, Theorem 5.2]: if $\mathcal{F}$ is a coherent sheaf on a projective A-scheme X, then the A-modules $\mathrm{H}^*(X, \mathcal{F})$ are finitely generated and $\mathcal{F}(n)$ is acyclic for $n \gg$

0. Recall the definition of *ample* and *very ample* sheaves [Har97, Ch. II, §7] and explain the implication (i) ⇒ (ii) of [Har97, Proposition 5.3], which generalizes the previous theorem.

(11) **Ext groups and sheaves (01.07)** Define Ext-groups and Ext-sheaves and summarize their basic properties [Har97, Propositions 6.2–6.8]. Explain [Har97, Proposition 6.9], which is a consequence of Serre vanishing.

(12) **Serre duality I: projective spaces and dualizing sheaves (08.07)** Prove Serre duality for $\mathbb{P}^n_k$ [Har97, Theorem 7.1]. Define the notion of *dualizing sheaf* and show that any projective scheme over a field admits a dualizing sheaf [Har97, Proposition 7.5].

(13) **Serre duality II: Cohen-Macaulay schemes and local complete intersections (15.07)** Prove Serre duality for projective Cohen–Macaulay schemes over algebraically closed fields [Har97, Theorem 7.6(b), (i) ⇒ (iii)]. Beforehand, recall the definition of Cohen–Macaulay local rings (e.g., [Har97, Ch. II, §8]) and the relevant properties used in the proof. In the case of a local complete intersection $X \subset \mathbb{P}^n_k$, sketch the identification of the dualizing sheaf in terms of the cotangent bundle of $\mathbb{P}^n_k$ and the conormal bundle of the embedding [Har97, Proposition 7.11]. When X is smooth, this says that the dualizing sheaf is the top exterior power of $\Omega_{X/k}$.

References

[Har97] R. Hartshorne, *Algebraic Geometry*, Springer, 1997

[Hoy] M. Hoyois, *Kommutative Algebra*, Vorlesung SS23, `https://hoyois.app.uni-regensburg.de/SS23/kommalg/skript.pdf`

[HS97] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer, 1997

[Stacks] The Stacks Project Authors, *The Stacks Project*, `http://stacks.math.columbia.edu`

[Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambrige University Press, 1994