WiSe 20/21 Algebraic Topology I
Exercise sheet 1 (due November 11)

Exercise 1.1.

(a) Let
$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots$$
be a sequence of topological spaces and continuous maps, let X_∞ be its colimit in Set, and let $i_n \colon X_n \to X_\infty$ be the canonical map. Define a topology on X_∞ as follows: a subset $U \subset X_\infty$ is open if and only if $i_n^{-1}(U)$ is open in X_n for all $n \geq 0$. With this topology, show that X is a colimit of the sequence $(X_n)_{n\geq 0}$ in the category Top.

(b) The infinite sphere S^∞ is defined as the colimit $\operatorname{colim}_{n\geq 0} S^n$, where S^n is identified with the subspace of S^{n+1} where the last coordinate is zero. Show that S^∞ is contractible.

Exercise 1.2. Show that the canonical functor $\mathsf{Top} \to \mathsf{hTop}$ preserves products and coproducts (indexed by arbitrary sets). In particular, the homotopy category hTop admits products and coproducts.

Exercise 1.3. A continuous map $f \colon X \to Y$ is called *constant* if there exists $y \in Y$ such that $f(x) = y$ for all $x \in X$. It is called *null-homotopic* if it is homotopic to a constant map.

Let X be a topological space. Prove that the following assertions are equivalent:

(a) X is contractible, i.e., $X \simeq *$.

(b) The identity map id_X is null-homotopic.

(c) For every $Y \in \mathsf{Top}$, every continuous map $Y \to X$ is null-homotopic.

(d) For every $Y \in \mathsf{Top}$, every continuous map $X \to Y$ is null-homotopic.

(e) X is nonempty, and for every $x \in X$, id_X is homotopic to the constant map with value x.