## Notation

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ $\quad (0 \in \mathbb{N})$
- $D^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$ n-disk
- $S^n = \partial D^{n+1} = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$ n-sphere
- $I = [0,1]$ unit interval
- $A \subset B$ set inclusion (does not imply $A \neq B$ !)
- $A - B$ set difference
- $A \hookrightarrow B$ injective map, $A \twoheadrightarrow B$ surjective map

## Category theory

- $Ob(C)$ set/class of objects of a category $C$
- $\mathrm{Hom}_C(X,Y)$ or $\mathrm{Hom}(X,Y)$ set of morphisms from $X$ to $Y$
- $\cong$ isomorphism
- $\mathrm{Fun}(C,D)$ category of functors from $C$ to $D$
- Set sets
  Top topological spaces
  Grp groups
  Ab abelian groups
  $\mathrm{Mod}_R$ R-modules

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## §0. Introduction

Algebraic topology aims to study topological spaces using algebraic invariants.

- Discrete invariants are functions

$$F: \ Ob(\mathrm{Top}) \longrightarrow E$$

such that if $X$ is homeomorphic to $Y$ then $F(X) = F(Y)$.

*Examples*:
- Euler characteristic $\chi : \{\text{finite CW-complexes}\} \longrightarrow \mathbb{Z}$

$$\chi(S^1) = 0, \ \chi(S^2) = 2.$$

- the Betti numbers $b_i : \{\text{finite CW-complexes}\} \longrightarrow \mathbb{N}, \ i \geq 0$

$$(n \neq 0) \quad b_i(S^n) = \begin{cases} 1 & \text{if } i = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$

- Categorical invariants are functors

$$F : \mathrm{Top} \longrightarrow \mathcal{C}, \quad \mathcal{C} \text{ a category}$$

often: $\mathcal{C} = \mathrm{Grp}, \mathrm{Ab}, \mathrm{Mod}_R, \ldots$

<u>Examples</u>

- connected components $\pi_0 : \mathrm{Top} \to \mathrm{Set}$
- the fundamental group / the first homotopy group

$$\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$$

- higher homotopy groups

$$\pi_n : \mathrm{Top}_* \longrightarrow \mathrm{Ab} \qquad (n \geq 2)$$

- homology groups $H_i : \mathrm{Top} \longrightarrow \mathrm{Ab} \quad (i \geq 0)$
- cohomology groups $H^i : \mathrm{Top}^{op} \longrightarrow \mathrm{Ab} \quad (i \geq 0)$
- "generalized cohomology theories": K-theory $K^i$, cobordism $\Omega^i$

- Higher categorical invariants: $F : \mathrm{Top} \longrightarrow \mathcal{C}$, $\mathcal{C}$ is an $n$-category $\infty \geq n \geq 2$.
  - fundamental $n$-groupoid.
  - shape theory.

<u>Sample of applications</u>

- the fundamental theorem of algebra (every non-constant $p(z) \in \mathbb{C}[z]$ has a zero in $\mathbb{C}$)

- Brouwer fixed-point theorem: every continuous map $f : D^n \longrightarrow D^n$ has a fixed point (i.e. $\exists\, x \in D^n$ such that $f(x) = x$).

- $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$, $S^n \not\cong S^m$ if $n \neq m$, $D^n \not\cong D^m$ ...
- Jordan curve theorem: if $C \subset \mathbb{R}^2$, $C \cong S^1$, then $\mathbb{R}^2 \setminus C$ is disconnected.


- Hedgehog Theorem: every continuous vector field on $S^2$ has a zero.

$$\begin{tikzcd} TS^2 \arrow[d] \\ S^2 \arrow[u, bend right, "\text{vector fields}"'] \end{tikzcd}$$

- Nielsen–Schreier theorem: if $G$ is a free group and $H \leq G$ is a subgroup, then $H$ is free.

- If $D$ is a finite-dimensional division algebra over $\mathbb{R}$, then $\dim_{\mathbb{R}} D \in \{0,1,2,4,8\}$

- If $n \geq 2$, $\mathbb{RP}^n$ does not embed smoothly into $\mathbb{R}^{n+1}$.

<u>Remark</u> Theorems such as $\mathbb{R}^n \not\cong \mathbb{R}^m$ are stronger than they appear, because continuity is a very weak condition.

$[0,1] \longrightarrow \mathbb{R}^2$

For example, there exists a surjective continuous map

$$D^1 \longrightarrow D^n \qquad \text{for all } n.$$

(called "space-filling curve")

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## §1. Homotopy

Recall: $I = [0,1]$ unit interval.

<u>Definition</u> • Let $f, g : X \longrightarrow Y$ be continuous maps. A <u>homotopy</u> from $f$ to $g$ is a continuous map $H : X \times I \longrightarrow Y$ such that $H(-,0) = f$ $\quad H(x,0) = f(x)\ \forall x \in X$

$H(-,1) = g$:

$i_t : X \hookrightarrow X \times I$
$x \longmapsto (x,t)$

$$\begin{tikzcd}
X \arrow[d, hook, "i_0"'] \arrow[dr, "f"] & \\
X \times I \arrow[r, dashed, "H"] & Y \\
X \arrow[u, hook, "i_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

- We say that $f$ and $g$ are <u>homotopic</u>, written $f \simeq g$, if there exists a homotopy from $f$ to $g$.

<u>Example</u>: $H : \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n$, $H(x,t) = tx$ is a homotopy from the constant map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ with value $0$ to the identity on $\mathbb{R}^n$.

<u>Properties</u> $X, Y, Z \in \mathrm{Top}$.

1) $\simeq$ is an equivalence relation on $\mathrm{Hom}_{\mathrm{Top}}(X,Y)$:
   - $f \simeq f$
   - $f \simeq g \Rightarrow g \simeq f$
   - $f \simeq g$ and $g \simeq h \Rightarrow f \simeq h$.

2) Let $f, g : X \longrightarrow Y$ be such that $f \simeq g$.
   - for any $h : Y \longrightarrow Z$, $h \circ f \simeq h \circ g$
   - for any $h : Z \longrightarrow X$, $f \circ h \simeq g \circ h$

Proof

1) • reflexivity: $H : X \times I \longrightarrow Y$, $(x,t) \mapsto f(x)$ is a homotopy from $f$ to $f$.

• symmetry: if $H : X \times I \longrightarrow Y$ is a homotopy from $f$ to $g$ then

$$X \times I \longrightarrow Y, \quad (x,t) \mapsto H(x, 1-t)$$

is a homotopy from $g$ to $f$.

• transitivity: let $H : X \times I \longrightarrow Y$ be a homotopy from $f$ to $g$, $K : X \times I \longrightarrow Y$ be a homotopy from $g$ to $h$.

Then

$$X \times I \longrightarrow Y$$
$$(x,t) \longmapsto \begin{cases} H(x, 2t) & \text{if } t \in [0, \tfrac{1}{2}] \\ K(x, 2t-1) & \text{if } t \in [\tfrac{1}{2}, 1] \end{cases}$$

is a homotopy from $f$ to $h$.

2) Let $H : X \times I \longrightarrow Y$ be a homotopy from $f$ to $g$.

• $h \circ H : X \times I \longrightarrow Z$ is a homotopy from $h \circ f$ to $h \circ g$.

• $H \circ (h \times id_I) : Z \times I \longrightarrow Y$ is a homotopy from $f \circ h$ to $g \circ h$. $\square$

$$Z \times I \xrightarrow{h \times id_I} X \times I \xrightarrow{H} Y$$

Notation: $[X, Y] = \mathrm{Hom}_{Top}(X, Y)/\simeq$ homotopy classes of continuous maps from $X$ to $Y$.

The homotopy category

hTop: objects: topological spaces

morphisms: $\mathrm{Hom}_{hTop}(X, Y) = [X, Y]$

composition: $[f] \circ [g] = [f \circ g]$ (well-defined by 2) above).