fib_b is essentially surjective

Let $G = \pi_1(B,b)$. Every G-set is a coproduct of G-sets of the form G/H.
Since fib_b preserves coproducts, it suffices to construct a covering p s.t. $\mathrm{fib}_b(p) \cong G/H$.
We do the case $H = \{e\}$ first.

Let $\tilde{B} = \{\alpha : I \to B \mid \alpha(0) = b\}/\simeq_p$, $\quad ev_1 : \tilde{B} \to B$, $ev_1([\alpha]_p) = \alpha(1)$.

Call an open $U \subset B$ good if it is path-connected and $\pi_1(U,x) \to \pi_1(B,x)$ is trivial for some $x \in U$ ($\Rightarrow$ any two paths in U are path-homotopic in B)
By $(\dagger)$, good open sets form a basis for the topology on B.
For U good, $\alpha : I \to B$, $\alpha(0) = b$, $\alpha(1) \in U$, we define

$$U(\alpha) = \{[\alpha * \beta]_p \mid \beta : I \to U\} \subset \tilde{B}.$$

The sets $U(\alpha)$ form a basis for a topology on $\tilde{B}$:

if $[\gamma]_p \in U(\alpha) \cap U'(\alpha')$
choose $\gamma(1) \in V \subset U \cap U'$ with V good.
Then $[\gamma]_p \in V(\gamma)$ and $V(\gamma) \subset U(\alpha) \cap U'(\alpha')$

$$\left(\gamma \simeq_p \alpha * \underbrace{\beta}_{\text{in } U} \Rightarrow \gamma * \underbrace{\delta}_{\text{in } V} \simeq_p \alpha * \underbrace{(\beta * \delta)}_{\text{in } U}\right)$$

Note that $ev_1 : \tilde{B} \to B$ is continuous and open:

$$ev_1^{-1}(U) = \bigcup_\alpha U(\alpha), \quad ev_1(U(\alpha)) = U.$$

If U is good and $x \in U$, we have a bijection

$$\begin{aligned} ev_1^{-1}(U) = \tilde{B} \times_B U &\longleftarrow \mathrm{Hom}_{\pi_1 B}(b,x) \times U \\ [\alpha * \alpha_y]_p &\longleftarrow\!\shortmid ([\alpha]_p, y) \end{aligned}$$

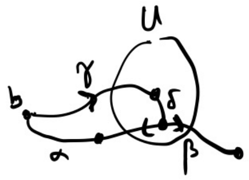
where α_y is any path from x to y in U

Under this bijection, $\{[\alpha]_p\} \times U$ corresponds to $U(\alpha)$ homeomorphically, so this bijection is in fact a homeomorphism.

$\Rightarrow$ $ev_1 : \tilde{B} \to B$ is a covering map with $\tilde{B}_x = \mathrm{Hom}_{\pi_1 B}(b,x)$.
Let $\beta : I \to B$ be a path from x to y.
<u>Claim</u>: $\beta_* : \tilde{B}_x \to \tilde{B}_y$ is given by $[\alpha]_p \mapsto [\beta]_p \circ [\alpha]_p$

Indeed, $\beta_*([\alpha]_p) = \tilde{\beta}(1)$ where $\tilde{\beta} : I \to \tilde{B}$ is the unique lift of β with $\tilde{\beta}(0) = [\alpha]_p$.

$$\begin{aligned} \tilde{\beta} : I &\longrightarrow \tilde{B} \\ s &\longmapsto [\alpha * \beta_s]_p \quad \text{where } \beta_s(t) = \beta(st) \\ 0 &\longmapsto [\alpha]_p \\ 1 &\longmapsto [\alpha * \beta]_p \end{aligned}$$



$\tilde{\beta}$ is continuous: if $[\alpha * \beta_s] \in U(\gamma)$, $\exists\, \delta : I \to U$ s.t. $[\alpha * \beta_s] = [\gamma * \delta]_p$.
If $\varepsilon > 0$ s.t. $\beta(B(s,\varepsilon)) \subset U$, then $\tilde{\beta}(B(s,\varepsilon)) \subset U(\gamma)$.

In particular, the monodromy action of $\pi_1(B,b)$ on $\tilde{B}_b = \pi_1(B,b)$ is the action of π_1 on itself.

Now if $H \leq G$, consider the map $\tilde{B}/H \xrightarrow{ev_1} B$ (where G acts on $\tilde{B}$ by $[\gamma]_p \cdot [\alpha]_p = [\gamma * \alpha]_p$).
Then $\tilde{B}/H \times_B U \cong \mathrm{Hom}_{\Pi_1 B}(L, x)/H \times U \Rightarrow \tilde{B}/H$ is a covering of B and $(\tilde{B}/H)_b \cong G/H$ as G-sets. $\square$

Analogies with Galois theory

base space B, connected and $(+)$	base field K
connected covering $E \to B$	finite separable field extension $K \subset L$ ($\mathrm{Spec}\, L \to \mathrm{Spec}\, K$)
Galois covering $E \to B$, $\mathrm{Aut}_B(E)$	finite Galois extension $K \subset L$, $\mathrm{Gal}(L/K)$
general covering $E \to B$	étale K-scheme ($\coprod_{i \in I} \mathrm{Spec}\, L_i \to \mathrm{Spec}\, K$)
B simply connected	K separably closed
$\Pi_1(B)$	groupoid of separable closures of K
choice of base point $b \in B$	choice of separable closure $K \subset K^{sep}$.
fundamental group $\pi_1(B,b)$	absolute Galois group $\mathrm{Gal}(K^{sep}/K)$.
fiber of $E \to B$ over b	set of embeddings $L \hookrightarrow K^{sep}$ (that fix K)

Proposition Suppose X is simply path-connected. Let G be a group acting on X on the left. Suppose that every $x \in X$ has a nhood U s.t. $gU \cap U = \emptyset$ for $g \neq e$. Then $\pi_1(X/G) \cong G$.

Proof Choose $x \in X$. By Exercise 5.4, $X \xrightarrow{p} X/G$ is a covering.

$$[\alpha]_p \longmapsto \alpha_*(x)$$

$$\begin{tikzcd}
\pi_1(X/G, p(x)) \arrow[r, "\cong"] \arrow[dr, dashed, "\varphi"'] & Gx & g^{-1}x \\
 & G \arrow[u, "\cong"] & g \arrow[u, mapsto]
\end{tikzcd}$$

Since we have a functor $\mathrm{Fib}_{p(x)} : \mathrm{Cov}_{X/G} \longrightarrow \mathrm{Set}_{\pi_1(X/G, p(x))}$, the action of G on Gx is π_1-equivariant: $g\alpha_*(x) = \alpha_*(gx)$.

Claim $\varphi : \pi_1(X/G, p(x)) \longrightarrow G$ is a group homomorphism

if $\alpha_*(x) = g^{-1}x$ $\quad \beta_*(x) = h^{-1}x$

then $\quad \alpha_*\beta_*(x) = \alpha_*(h^{-1}x) = h^{-1}\alpha_*(x) = h^{-1}g^{-1}x = (gh)^{-1}x$ $\qquad \square$.

Examples
- $\mathbb{Z} \circlearrowright \mathbb{R}$ by $n \cdot r = r + n$, $\mathbb{R}/\mathbb{Z} \cong S^1 \Rightarrow \pi_1(S^1) \cong \mathbb{Z}$
- C_2 acts on S^n by $x \mapsto -x$, $S^n/C_2 \cong \mathbb{R}P^n \Rightarrow \pi_1(\mathbb{R}P^n) \cong C_2$, $n \geq 2$.

Remarks. 1) If $n \geq 2$, S^n and $S^n \times I$ are simply connected. Hence if $p : E \to B$ is a covering, then $p_* : \pi_n(E, e) \longrightarrow \pi_n(B, b)$ is an isomorphism.

For example, $\pi_n(S^1) = 0$ for $n \geq 2$, because there is a covering $\mathbb{R} \to S^1$.

2) If $p : E \to B$ is a locally trivial bundle, $e \in E$, $b = p(e)$, then is a long exact sequence

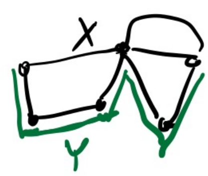
$$\cdots \to \pi_{n+1}(B, b) \to \pi_n(E_b, e) \to \pi_n(E, e) \to \pi_n(B, b) \to \pi_{n-1}(E_b, e) \to \cdots$$

eg: $S^3 \to S^2$ Hopf fibration $\Rightarrow \pi_n(S^3) \xrightarrow{\cong} \pi_n(S^2)$ for $n \geq 3$.

Application: the Nielsen–Schreier theorem

Lemma Let X be obtained from a discrete space X_0 by attaching 1-cells. (X is a "graph"). Then $\pi_1(X, x)$ is free for all $x \in X$.

Proof. WLOG we can assume X connected. Fix $x_0 \in X_0$. Any connected graph has a spanning tree $Y \subset X$



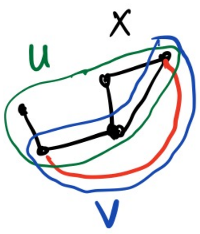
For each $x \in X_0$, we choose a path γ_x from x_0 to x.

<u>Claim</u> $\pi_1(X, x_0)$ is free on the set $\left\{ \gamma_{\sigma(0)} * \sigma * \overline{\gamma_{\sigma(1)}} \;\middle|\; \begin{array}{l} \sigma : I \to X \text{ ranges over} \\ \text{the 1-cells in } X \text{ not in } Y \end{array} \right\}$

To prove this, we can assume X has finitely many 0-cells and 1-cells (using that π_1 commutes with filtered colimits of open subspaces).

A finite tree is contractible $\Rightarrow$ $\pi_1(Y) = \{e\}$.

If X is a connected graph and X' is obtained from X by attaching a single 1-cell, then $\pi_1(X') \cong \pi_1(X) * \mathbb{Z}$ (by SvK theorem)



$U \simeq X$

$V \simeq S^1$

$U \cap V \simeq *$

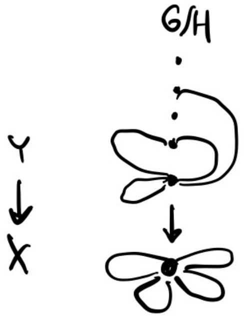
This concludes the proof by induction. $\square$

<u>Theorem</u> (Nielsen–Schreier)

If G is a free group and $H \leq G$ is a subgroup, then H is free.

<u>Proof</u> Suppose G is free on a set I. Let $X = \bigvee_{i \in I} S^1$, so that $\pi_1(X) \cong G$.

Let $Y \xrightarrow{p} X$ be a connected covering with $\mathrm{fib}_*(p) = G/H \circlearrowleft G$.



$\Rightarrow$ Y is obtained from G/H by attaching 1-cells.

$\Rightarrow$ by the lemma $H = \pi_1(Y)$ is free. $\square$