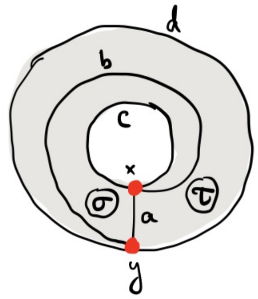
§5. Simplicial sets

Idea: A simplicial set is a combinatorial model for a topological space with a triangulation.

Example

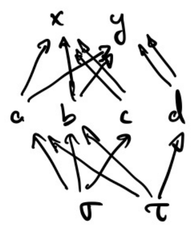
Annulus:



2 vertices:

4 edges:

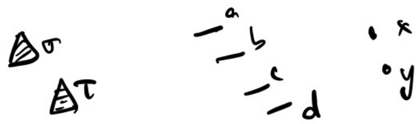
2 faces:



$$\begin{array}{c} X_0 \\ \uparrow\uparrow \\ X_1 \\ \uparrow\uparrow\uparrow \\ X_2 \end{array}$$

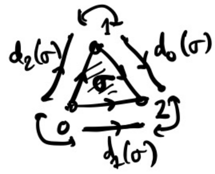
We can reconstruct the annulus up to homeomorphism from the diagram of sets $X_2 \overset{d_0}{\underset{d_2}{\rightrightarrows}}{}^{d_1} X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0$:

$$\left(X_2 \times \Delta^2 \amalg X_1 \times \Delta^1 \amalg X_0 \times \Delta^0 \right) / \sim$$



Not every diagram $X_2 \rightrightarrows X_1 \rightrightarrows X_0$ correspond to a triangulation:

d_i = face opposite the i-th vertex.



$$\left.\begin{array}{l} d_0 d_0(\sigma) = d_0 d_1(\sigma) \\ d_0 d_2(\sigma) = d_1 d_0(\sigma) \\ d_1 d_1(\sigma) = d_1 d_2(\sigma) \end{array}\right\} \text{"simplicial identities".}$$

Definition Let C be a category.

- A semi-simplicial object X in C is a collection of objects X_n, $n \geq 0$, and maps $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, called face maps satisfying

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{for } i < j.$$

- A simplicial object X in C is a collection of objects X_n, $n \geq 0$, and maps

$d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$ (face maps)

$s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$ (degeneracy maps)

satisfying the simplicial identities:

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{for } i<j$$
$$s_i \circ s_j = s_j \circ s_{i-1} \quad \text{for } i>j$$
$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i<j \\ \mathrm{id} & \text{if } i=j \text{ or } i=j+1 \\ s_j \circ d_{i-1} & \text{if } i>j+1. \end{cases}$$

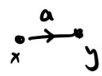
- If $C = \mathrm{Set}$, the elements of X_n are called the <u>n-simplices</u> of X (and 0-simplex $\equiv$ vertex, 1-simplex $\equiv$ edge)
- An n-simplex is called <u>degenerate</u> if it is in the image of s_i for some i.

<u>Notation</u>: sC is the category of simplicial objects in C.

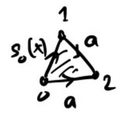
Interpretation: d_i (n-simplex σ) = (n−1)-simplex opposite the i-th vertex of σ.

s_i (n-simplex σ) = (n+1)-simplex squashed to an n-simplex along the edge $i \to i+1$

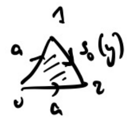
e.g.:



$d_1(a) = x$
$d_0(a) = y$

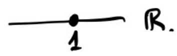
$s_0(a) =$ 

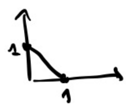
$$\begin{pmatrix} d_2 s_0(a) = s_0 d_1(a) \\ d_0 s_0(a) = d_1 s_0(a) = a. \end{pmatrix}$$

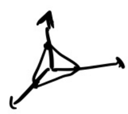
$s_1(a) =$ 

<u>Geometric realization</u>

<u>Def</u>. The <u>topological n-simplex</u> is $\Delta^n = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0 \text{ and } \sum_{i=0}^n t_i = 1 \} \subset \mathbb{R}^{n+1}$

Δ^0: 

Δ^1: 

Δ^2: 

We have maps $\delta_i : \Delta^{n-1} \hookrightarrow \Delta^n$, $0 \leq i \leq n$ — inclusion of the face opposite the i-th vertex

$$\delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, \underset{\substack{\uparrow \\ i\text{-th position.}}}{0}, \dots, t_{n-1})$$

$\sigma_i : \Delta^{n+1} \twoheadrightarrow \Delta^n$, $0 \leq i \leq n$ — squash the edge $i \to i+1$.

$$\sigma_i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1})$$

Definition

- If X is a semi-simplicial set, we define $\|X\| \in \mathrm{Top}$ by
$$\|X\| = \left(\coprod_{n\geq 0} X_n \times \Delta^n\right) \Big/ (d_i x, u) \sim (x, \delta_i(u))$$

- If X is a simplicial set, its geometric realization $|X| \in \mathrm{Top}$
$$|X| = \left(\coprod_{n\geq 0} X_n \times \Delta^n\right) \Big/ \begin{array}{l} (d_i x, u) \sim (x, \delta_i u), \\ (s_i x, u) \sim (x, \sigma_i u) \end{array}$$

Remark: If X is a simplicial set, there is a quotient map $\|X\| \twoheadrightarrow |X|$.
If $X \neq \emptyset$, this map is not a homeomorphism, but one can show that it is a homotopy equivalence. $\|X\|$ is sometimes called the fat geometric realization.

Simplicial abelian group and chain complexes

- A chain complex of abelian groups C_* is a sequence of abelian groups
$$\cdots \to C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \qquad \text{such that } d\circ d = 0$$

$$Z_i(C_*) = \ker(d : C_i \to C_{i-1}). \qquad \text{"cycles"}$$
$$B_i(C_*) = \mathrm{im}(d : C_{i+1} \to C_i) \qquad \text{"boundaries"}$$
$$H_i(C_*) = Z_i(C_*)/B_i(C_*) \qquad i\text{-th homology group of } C_*$$

- A chain map $f : C_* \to D_*$ is a collection of maps $f_n : C_n \to D_n$ such that $f_n \circ d = d \circ f_{n+1}$.

We write $\mathrm{Ch}(\mathrm{Ab})$ for the category of chain complexes and chain maps.
$\mathrm{Ch}_{\geq 0}(\mathrm{Ab}) \subset \mathrm{Ch}(\mathrm{Ab})$ full subcategory where $C_i = 0$ for $i < 0$.

More generally, we can replace Ab by any abelian category, e.g., Mod_R.

Definition: Let A be a semi-simplicial abelian group.

1) The Moore complex of A is the chain complex
$$C_*A : \qquad C_nA = A_n$$
$$d : A_n \to A_{n-1} \text{ is } \sum_{i=0}^{n} (-1)^i d_i$$

2) The normalized complex of A is the chain complex
$$N_*A : \qquad N_nA = \bigcap_{i=1}^{n} \ker(d_i : A_n \to A_{n-1}) \subset A_n$$
$$d = d_0.$$

Lemma These are well-defined chain complexes.

Proof: 1) $A_n \xrightarrow{d} A_{n-1} \xrightarrow{d} A_{n-2}$

$$d^2 = \sum_{i=0}^{n-1} (-1)^i d_i \left(\sum_{j=0}^{n} (-1)^j d_j \right) = \sum_{i,j} (-1)^{i+j} d_i d_j \qquad \left[d_i d_j = d_{j-1} d_i \text{ for } i<j \right]$$

$$= \underbrace{\sum_{i<j} (-1)^{i+j} d_{j-1} d_i}_{\substack{\parallel\ k=j-1,\ l=i \\ \sum_{l \le k} (-1)^{k+l+1} d_k d_l}} + \sum_{j \le i} (-1)^{i+j} d_i d_j = 0.$$

2) $\alpha \in N_n A$

$$\underset{0 \le i \le n-1}{d_i d_0(\alpha)} = d_0 d_{i+1}(\alpha) \underset{\alpha \in N_n A}{=} 0 \implies \begin{array}{l} d_0(\alpha) \in N_{n-1}A. \\ d_0^2(\alpha) = 0 \end{array} \qquad \square.$$

Definition: Let X be a semi-simplicial set and A an abelian group. We let

$$C_*(X, A) := C_*(\mathbb{Z}[X] \otimes A) \qquad \text{where } \mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Ab}, \quad S \mapsto \bigoplus_S \mathbb{Z}.$$

The <u>homology of X with coefficients in A</u> is the homology of $C_*(X, A)$:

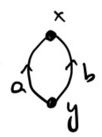
$$H_*(X, A) := H_*(C_*(X, A)).$$

If $A = \mathbb{Z}$, we also write $C_*(X) = C_*(X, \mathbb{Z})$ and $H_*(X) = H_*(X, \mathbb{Z})$.

Example

$X_0 = \{x, y\}$ $\quad d_0 : a, b \mapsto x$

$X_1 = \{a, b\}$ $\quad d_1 : a, b \mapsto y$

$\|X\| =$ 

$\cong S^1.$

$$C_*(X) : \quad \mathbb{Z}a \oplus \mathbb{Z}b \xrightarrow{d} \mathbb{Z}x \oplus \mathbb{Z}y$$

$$\begin{array}{l} d(a) = x - y \\ d(b) = x - y \end{array} \implies \ker(d) = \mathbb{Z}(a-b) \qquad \mathrm{im}(d) = \mathbb{Z}(x-y)$$

$$H_0(X) = \frac{\mathbb{Z}x \oplus \mathbb{Z}y}{\mathrm{im}(d)} \cong \mathbb{Z}$$

$$H_1(X) = \frac{\ker(d)}{0} \cong \mathbb{Z}$$

$$H_n(X) = 0 \quad \text{for} \quad n \ne 0, 1$$