Definition.
- A chain complex C_* is <u>acyclic</u> if $H_*(C_*) = 0$.
- A chain map $f: C_* \to D_*$ is a <u>quasi-isomorphism</u> if it induces an isomorphism $H_*(C_*) \xrightarrow{\cong} H_*(D_*)$.

Proposition A short exact sequence of chain complexes

$$0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0.$$

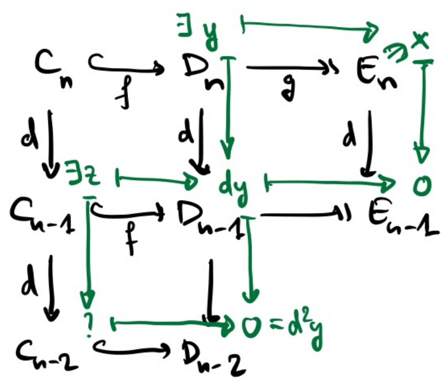
gives rise to a long exact sequence

$$\cdots \to H_n(C_*) \xrightarrow{f_*} H_n(D_*) \xrightarrow{g_*} H_n(E_*) \xrightarrow{\partial} H_{n-1}(C_*) \to \cdots$$

where $\partial([x]) = [\text{"} f^{-1} d\, g^{-1}(x) \text{"}]$ for $x \in Z_n(E_*)$.

Corollary: If $f: C_* \to D_*$ is degreewise injective (resp. surjective), then f is a quasi-isomorphism if and only if $\mathrm{coker}(f)$ (resp. $\ker(f)$) is acyclic.

Proof. ∂ is <u>well-defined</u>:



$\partial([x]) = [z]$

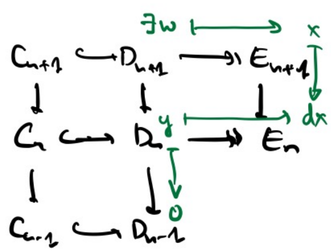
If y' is another preimage of x, then $g(y-y') = 0$.

$\Rightarrow \exists\, w \in C_n$ such that $f(w) = y - y'$

$\Rightarrow dw = z - z'$, so z is well-defined modulo $B_{n-1}(C_*)$

<u>Im $\subset$ ker</u>:
- $g_* \circ f_* = 0$ since $g \circ f = 0$.
- if $y \in Z_n(D_*)$, then $dy = 0$ so $z = 0 \Rightarrow \partial \circ g_* = 0$
- $\underset{\shortparallel\, dy}{f(z)} \in B_{n-1}(D_*) \Rightarrow f_* \circ \partial = 0$

<u>ker $\subset$ im</u>:
- if $\partial([x]) = 0$ then $z = dw$, $d(y - f(w)) = 0$, $g(y - f(w)) = g(y) = x$ so $[x] = g_*[y - f(w)]$.
- $[z] \in \ker(f_*)$, then $f(z) \in B_{n-1}(D_*)$ so $\exists\, y \in D_n$ s.t. $dy = f(z)$ then $[z] = \partial([g(y)])$



- if $g_*([y]) = 0 \Rightarrow g(y) \in B_n(E) \Rightarrow g(y) = dx$ for some $x \in E_{n+1}$.
  let $w \in D_{n+1}$ s.t. $g(w) = x$.
  $g(y - dw) = 0 \Rightarrow \exists\, z \in C_n$ st $f(z) = y - dw$
  $f(dz) = d(f(z)) = dy - d^2 w = 0 \Rightarrow dz = 0$ so $f_*[z] = [y]$. $\square$

Let A be a simplicial abelian group. Let $D_n A \subset A_n$ be the subgroup generated by the degenerate n-simplices in A.

Lemma. $D_* A$ is a subcomplex of $C_* A$, i.e. $d: A_n \to A_{n-1}$ sends $D_n A$ to $D_{n-1} A$.

Proof. It suffices to show $d(s_i(x)) \in D_{n-1} A$ for all $x \in A_{n-1}$, $0 \le i \le n-1$.

$$d(s_i(x)) = \sum_{j=0}^{n} (-1)^j d_j s_i(x) \overset{\text{simp. id.}}{=} \sum_{j > i+1} s_i(\cdots) + \sum_{j<i} s_{i-1}(\cdots) + \underbrace{(-1)^{i-1} x + (-1)^i x}_{=0} \in D_{n-1} A.$$

$\square$.

Theorem. Let A be a simplicial abelian group.

1) The inclusion $N_* A \subset C_* A$ is a quasi-isomorphism

2) The composition $N_* A \hookrightarrow C_* A \twoheadrightarrow C_* A / D_* A$ is an isomorphism.

Proof. We prove: 1) $D_* A$ is acyclic

2) $N_* A \hookrightarrow C_* A \twoheadrightarrow C_* A / D_* A$ is an isomorphism

1) Let $F_p D_n \subset D_n A$ be the subgroup generated by $s_i(A_{n-1})$ for $0 \le i \le p$.

so $F_{n-1} D_n = D_n A$.

Then $F_p D_*$ is a subcomplex of $D_* A$ and we have a filtration

$$0 = F_{-1} D_* \subset \cdots \cdots \subset F_{p-1} D_* \subset F_p D_* \subset \cdots \subset D_* A = \bigcup_{p \ge 0} F_p D_* .$$

It suffices to show that each $F_{p-1} D_* \hookrightarrow F_p D_*$ is a quasi-isomorphism, or that $gr_p D_* = F_p D_* / F_{p-1} D_*$ is acyclic,

$$\begin{array}{l} gr_p D_n = s_p(A_{n-1}) / s_p(A_{n-1}) \cap \langle s_i(A_{n-1}), i < p \rangle . \\ d \uparrow \\ gr_p D_{n+1} = s_p(A_n) / \cdots \end{array}$$

<u>Computation</u>: $x \in A_{n-1}$.

$$d_i s_p(x) = \begin{cases} s_{p-1} d_i(x) & \text{if } i < p \\ x & \text{if } i = p \text{ or } p+1 \\ s_p d_{i-1}(x) & \text{if } i > p+1. \end{cases}$$

$$(*) \quad \bullet \ d(s_p(x)) = \sum_{i=0}^{n} (-1)^i d_i s_p(x)$$

$$= \sum_{i=p+2}^{n} (-1)^i s_p d_{i-1}(x)$$

$$\bullet \ d s_p s_p(x) \overset{(*)}{=} \sum_{i=p+2}^{n+1} (-1)^i s_p \underbrace{d_{i-1} s_p}_{\substack{s_p d_{i-2} \text{ if } i-1 > p+1 \\ \text{id if } i-1 = p+1}}(x) = (-1)^{p+2} s_p(x) + \underbrace{\sum_{i=p+3}^{n+1} (-1)^i s_p s_p d_{i-2}(x)}_{\substack{= -s_p d s_p(x) \\ \text{by } (*)}}$$

So if $ds_p(x) = 0$ in $gr_p D_*$, then $s_p(x) = d(\pm s_p^2(x))$,

that is $\ker(d) \subset \operatorname{im}(d) \implies gr_p D_*$ is acyclic.

2) (sketch) We construct an inverse.

Let $r : A_n \longrightarrow A_n$ be $r_1 \circ r_2 \circ \cdots \circ r_n$

where $r_i : A_n \longrightarrow A_n$ is $id - s_{i-1} d_i$

Then one can check:

- $r(A_n) \subset N_n A$
- $r : C_* A \longrightarrow N_* A$ is a chain map.
- $r(D_n A) = 0$
- the reduced map $A_n / D_n A \xrightarrow{r} N_n A$ is inverse to the given map $N_n A \longrightarrow A_n / D_n A$. $\square$