Let X be a simplicial set. Then $X_n = X_n^{nd} \amalg X_n^{deg}$

$$\Rightarrow \quad \mathbb{Z}[X_n] = \mathbb{Z}[X_n^{nd}] \oplus \underbrace{\mathbb{Z}[X_n^{deg}]}_{D_n\mathbb{Z}[X]}$$

$$\Rightarrow \quad N_n\mathbb{Z}[X] \cong \mathbb{Z}[X_n]/_{D_n\mathbb{Z}[X]} \xleftarrow{\cong} \mathbb{Z}[X_n^{nd}]$$

Hence, the normalized complex $N_*\mathbb{Z}[X]$ is:

$$\cdots \longrightarrow \mathbb{Z}[X_n^{nd}] \xrightarrow{d} \mathbb{Z}[X_{n-1}^{nd}] \longrightarrow \cdots$$

with $d(x) = \sum_{i=0}^{n} (-1)^i \begin{cases} d_i(x) & \text{if non-deg.} \\ 0 & \text{otherwise} \end{cases}$ for $x \in X_n^{nd}$

<u>Theorem*</u> (Dold–Kan correspondence)

The functor $N : sAb \longrightarrow Ch_{\geq 0}(Ab)$ is an equivalence of categories.

<u>The functorial point of view</u>

<u>Def</u> The <u>simplex category</u> Δ has objects $[n] = \{0,1,\dots,n\}$, $n \geq 0$, and morphisms are non-decreasing maps.

- $\delta_i : [n-1] \hookrightarrow [n]$ $\quad 0 \leq i \leq n$
  is the injective map that misses $i \in [n]$: $\delta_i = (0,1,\dots,i-1,i+1,\dots,n)$
- $\sigma_i : [n+1] \twoheadrightarrow [n]$, $0 \leq i \leq n$
  is the surjective map that repeats i : $\sigma_i = (0,1,\dots,i,i,\dots,n)$
- Let $\Delta_{inj} \subset \Delta$ be the subcategory whose morphisms are strictly increasing.

<u>Definition</u> Let C be a category.

- A <u>semi-simplicial object</u> in C is a functor $X : \Delta_{inj}^{op} \longrightarrow C$
- A <u>simplicial object</u> in C is a functor $X : \Delta^{op} \longrightarrow C$.

<u>Notation:</u> for $\alpha : [m] \to [n]$, $X_n = X([n])$ and $\alpha^* = X(\alpha) : X_n \to X_m$

$d_i = \delta_i^*$, $s_i = \sigma_i^*$.

<u>Proposition</u> This definition is equivalent to the previous one.

<u>Proof.</u> • The maps δ_i and σ_i satisfy the dual of the simplicial identities
$\Rightarrow$ any $X : \Delta^{op} \to C$ gives rise to a simplicial object (X_n, d_i, s_i)

- In Δ, any map $\alpha : [m] \to [n]$ factors uniquely as

$$\begin{tikzcd} {[m]} \arrow[r] \arrow[dr, "\alpha_{surj}"'] & {[n]} \\ & {[p]} \arrow[u, "\alpha_{inj}"'] \end{tikzcd}$$

$$\alpha = \underbrace{\delta_{j_\ell} \cdots \delta_{j_1}}_{\alpha_{inj}} \underbrace{\sigma_{i_k} \cdots \sigma_{i_1}}_{\alpha_{surj}} \qquad \text{with} \qquad \begin{array}{l} i_1 \leq \cdots \leq i_k \\ j_1 \geq \cdots \geq j_\ell . \end{array}$$

Moreover, any sequence of δ_j's and σ_i's can be put in this form using the simplicial identities.

$\Rightarrow$ any (X_n, d_i, s_i) extends uniquely to a functor $X : \Delta^{op} \to C$. $\square$

Categorical digression: presheaves & Yoneda

If C is a small category, the category of presheaves (of sets) on C is

$$P(C) = \mathrm{Fun}(C^{op}, \mathrm{Set}).$$

The Yoneda embedding is the functor

$$\begin{aligned} C &\longrightarrow P(C) \\ X &\longmapsto \mathrm{Hom}_C(-, X) \end{aligned}$$

presheaf represented by X

Proposition

1) (Yoneda lemma) For any $F \in P(C)$, and $X \in C$, there is a bijection

$$\begin{aligned} F(X) &\cong \mathrm{Nat}(\mathrm{Hom}_C(-, X), F) \\ \varphi(\mathrm{id}_X) &\longleftarrow\!\shortmid\ \varphi \\ a &\longmapsto (Y \xrightarrow{f} X \mapsto f^*(a) \in F(Y)) \end{aligned}$$

In particular, the Yoneda embedding is fully faithful.

2) (Universal property of $P(C)$) Let D be a category with colimits. Then the Yoneda embedding $C \hookrightarrow P(C)$ induces an equivalence

$$\mathrm{Fun}^L(P(C), D) \xrightarrow{\simeq} \mathrm{Fun}(C, D)$$

where Fun^L = colimit-preserving functors. If $u : C \to D$, its unique colimit-preserving extension is

$$u_! : P(C) \to D, \quad u_!(F) = \operatorname*{colim}_{(X,a) \in El(F)} u(X)$$

$El(F)$: objects $(X \in C, a \in F(X))$
morphisms: $(X,a) \to (Y,b)$
$= f : X \to Y$ s.t. $f^*(b) = a$.
"category of elements of F"

Moreover, the functor $u_!$ has a right adjoint

$$u^*: D \longrightarrow P(C), \qquad u^*(Y) = \mathrm{Hom}_D(u(-), Y).$$

Geometric realization revisited

There is a functor
$$\begin{aligned} \Delta &\longrightarrow Top \\ [n] &\longmapsto \Delta^n \end{aligned}$$

$\alpha:[m]\to[n] \longmapsto \alpha_*: \Delta^m \to \Delta^n$ is the unique linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ sending the i-th vertex to the $\alpha(i)$-th vertex.

By the universal property of $P(-)$, we obtain

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, hook] & Top \\ P(\Delta) = sSet \arrow[ur, dashed, "\exists!\ |\cdot|"', "\text{colimit-preserving}"] & \end{tikzcd}$$

$X: \Delta^{op} \to Set$

$$|X| = \operatorname*{colim}_{([n],a)\in El(X)} \Delta^n = \coprod_{n\geq 0} \underbrace{\coprod_{X_n} \Delta^n}_{X_n \times \Delta^n} \Big/ \begin{array}{l}(\alpha^*(x), u) \\ \sim (x, \alpha_*(u))\end{array}$$

$\big[$ $([m],a) \to ([n],b)$ is $\alpha:[m]\to[n]$ s.t. $\alpha^*(b) = a$

Similarly, $\|\cdot\|$ is the unique colimit-preserving extension of $\Delta_{inj} \subset \Delta \longrightarrow Top$.

Examples:

- $\Delta[n] = \mathrm{Hom}_\Delta(-,[n])$. This is called the n-simplex.

often Δ^n

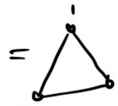
By definition, $|\Delta[n]| \cong \Delta^n$.

Yoneda lemma: $\mathrm{Hom}_{sSet}(\Delta[n], X) \cong X_n$

$$\Delta[n]^{nd}_k = \mathrm{Hom}_{\Delta_{inj}}([k],[n]) = \{\text{subsets of } \{0,\dots,n\} \text{ of size } k+1\}.$$

- $\partial\Delta[n] \subset \Delta[n]$ is defined by $\partial\Delta[n]_k = \{[k]\to[n] \text{ non-surjective}\}$

$$\partial\Delta[n]^{nd}_k = \begin{cases} \Delta[n]^{nd}_k & \text{if } k \neq n \\ \emptyset & \text{if } k \geq n \end{cases}$$

$$|\partial\Delta[n]| \cong \partial\Delta^n \cong S^{n-1}$$ e.g. $\partial\Delta^2 =$  $\cong S^1$.

- $$\begin{tikzcd} \partial\Delta[n] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "po"] & \Delta[n] \arrow[d] \\ * \arrow[r] & \Delta[n]/\partial\Delta[n] \end{tikzcd} \quad \overset{|\cdot|}{\rightsquigarrow} \quad \begin{tikzcd} \partial\Delta^n \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "po"] & \Delta^n \arrow[d] \\ * \arrow[r] & \Delta^n/\partial\Delta^n \cong S^n \end{tikzcd}$$

$$(\Delta[n]/\partial\Delta[n])^{nd}_k = \begin{cases} * & \text{if } k = 0, n \\ \emptyset & \text{otherwise.} \end{cases}$$
$(n \geq 1)$

$$|\Delta[2]/\partial\Delta[2]| = $$ 

- Let C be a small category. The <u>nerve</u> of C is the simplicial set

$$N(C) : \Delta^{op} \longrightarrow \mathrm{Set}, \quad [n] \longmapsto \mathrm{Hom}_{\mathrm{Cat}}(\underset{0\to 1\to\cdots\to n}{[n]}, C) \, .$$

$$N(C)_0 = \mathrm{Ob}(C)$$
$$N(C)_1 = \mathrm{Mor}(C)$$
$$N(C)_2 = \mathrm{Mor}(C) \underset{t,\ \mathrm{Ob}(C),\ s}{\times} \mathrm{Mor}(C)$$

The functor $N : \mathrm{Cat} \longrightarrow \mathrm{sSet}$ is fully faithful

<u>Remark.</u> $|\cdot|$ plays well with products, e.g.: $|\Delta[n] \times \Delta[m]| \xrightarrow{\cong} \Delta^n \times \Delta^m$

$\|\cdot\|$ does not: $\|\Delta[n]^{nd} \times \Delta[m]^{nd}\| \xrightarrow{\not\cong} \Delta^n \times \Delta^m$.