Simplicial homotopy.

Idea: replace I by $\Delta[1]$.

Definition • Let $f, g : X \to Y$ be morphisms in sSet. A (simplicial) homotopy from f to g is a morphism $H : X \times \Delta[1] \to Y$ such that:

$$\begin{tikzcd}
X \times \Delta[0] \cong X \arrow[d, hook, "id \times \delta_0"'] \arrow[dr, "f"] & \\
X \times \Delta[1] \arrow[r, "H"] & Y \\
X \times \Delta[0] \cong X \arrow[u, hook, "id \times \delta_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

• $f, g : X \to Y$ are (simplicially) homotopic if $\exists$ a sequence of morphisms $f = f_0, f_1, \dots, f_n = g$ and homotopies from f_{i-1} to f_i for all $1 \leq i \leq n$.

Remarks 1) The relation "$\exists$ a homotopy from f to g" is reflexive and symmetric, but not transitive:

$$\underset{1}{I} \amalg_{*} \underset{0}{I} \cong I \quad \text{but} \quad \underset{\delta_0}{\Delta[1]} \amalg_{*} \underset{\delta_1}{\Delta[1]} \ncong \Delta[1]$$

$$\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet \ \ncong \ \bullet\!-\!\!-\!\bullet$$

2) $|X \times \Delta[1]| \xrightarrow{\cong} |X| \times |\Delta[1]| \cong |X| \times I \Rightarrow |\cdot|$ preserves homotopies.
(Exercise 7.4)

Proposition If $f, g : X \to Y$ are simplicially homotopic, then $C_*(f), C_*(g) : C_*(X) \to C_*(Y)$ are chain-homotopic. In particular, $f_* = g_* : H_*(X, A) \to H_*(Y, A)$.

Proof. WLOG $\exists$ simplicial homotopy $H : X \times \Delta[1] \to Y$ from f to g.

Let $\theta_{n,i} : [n] \to [1]$ $\quad \theta_{n,i}(j) = \begin{cases} 0 & \text{if } j \leq i \\ 1 & \text{if } j > i \end{cases} \quad (-1 \leq i \leq n)$.

For $0 \leq i \leq n$, let $h_{n,i} : X_n \to Y_{n+1}$, $h_{n,i}(x) = H(s_i(x), \theta_{n+1,i})$.

Note: $d_j(\theta_{n,i}) = \begin{cases} \theta_{n-1,i-1} & \text{if } j \leq i \\ \theta_{n-1,i} & \text{if } j > i \end{cases}$

$$\Rightarrow \quad d_0 h_{n,0}(x) = H(\overbrace{d_0 s_0(x)}^{id}, \overbrace{\theta_{n,-1}}^{1}) = H(x, \delta_0(*)) = f(x).$$

$$d_{n+1} h_{n,n}(x) = H(\overbrace{d_{n+1} s_n(x)}^{id}, \overbrace{\theta_{n,n}}^{0}) = H(x, \delta_1(*)) = g(x)$$

$$d_i h_{n,j} = \dots = \begin{cases} h_{n-1,j-1} d_i & \text{if } i < j \\ d_i h_{n,j-1} & \text{if } i = j > 0 \\ h_{n-1,j} d_{i-1} & \text{if } i > j+1 \end{cases}$$

$\Rightarrow$ we have a semi-simplicial homotopy from f to g.
We conclude by Exercise 6.4. $\square$

§ 6. Singular homology

Definition Let X be a topological space.

- The singular simplicial set $\mathrm{Sing}(X)$ of X is defined by:

$$\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$$

$$\alpha : [m] \to [n], \quad \alpha^* : \mathrm{Sing}_n(X) \to \mathrm{Sing}_m(X), \ f \mapsto f \circ \alpha_*$$

- The singular chain complex of X with coefficients in an abelian group A is

$$C_*(X, A) = C_*(\mathrm{Sing}(X), A) = C_*(\mathbb{Z}[\mathrm{Sing}(X)] \otimes A).$$

- The (singular) homology of X with coefficients in A is

$$H_*(X, A) = H_*(C_*(X, A)).$$

Remark. By the universal property of $P(\Delta)$, $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$ is right adjoint to $|\cdot| : \mathrm{sSet} \to \mathrm{Top}$, i.e: for $X \in \mathrm{Top}$ and $S \in \mathrm{sSet}$,

$$\mathrm{Hom}_{\mathrm{Top}}(|S|, X) \cong \mathrm{Hom}_{\mathrm{sSet}}(S, \mathrm{Sing}(X)).$$

Variant (homology of pairs)

A pair of topological spaces (X, Y) is a topological space X with a subspace $Y \subset X$.

$$C_*(X, Y; A) = \mathrm{coker}(C_*(Y, A) \hookrightarrow C_*(X, A))$$
$$H_*(X, Y; A) = H_*(C_*(X, Y; A))$$

Remarks:
- $C_*(X) = C_*(X, \emptyset)$, $H_*(X) = H_*(X, \emptyset)$
- $C_*(X, X) \simeq 0$, $H_*(X, X) = 0$
- $C_*(X, Y, A) \cong C_*(X, Y) \otimes A$ (because $- \otimes A$ preserves cokernel)
- Warning: $H_*(X, Y) \not\cong \mathrm{coker}(H_*(Y) \to H_*(X))$ in general.

Functoriality: We have a sequence of functors

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "\mathrm{Sing}"] & \mathrm{sSet} \arrow[r, "\mathbb{Z}[-]"] & \mathrm{sAb} \arrow[r, "C_* \text{ or } N_*"] \arrow[loop below, "-\otimes A"] & \mathrm{Ch}_{\geq 0}(\mathrm{Ab}) \arrow[r, "H_n"] \arrow[loop below, "-\otimes A"] & \mathrm{Ab} \end{tikzcd}$$

For $f: X \to Y$, we write $f_*: C_*(X) \to C_*(Y)$, $f_*: H_*(X) \to H_*(Y)$.

- A morphism of pairs $f: (X,Y) \to (X',Y')$ is $f: X \to X'$ such that $f(Y) \subset Y'$.

$\rightsquigarrow$ category Pair of pairs

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, hook, "i'"] & X' \end{tikzcd} \quad \overset{C_*}{\rightsquigarrow} \quad \begin{tikzcd} C_*(Y) \arrow[r, hook, "i_*"] \arrow[d, "g_*"] & C_*(X) \arrow[r, two heads, "\mathrm{coker}"] \arrow[d, "f_*"] & C_*(X,Y) \arrow[d, dashed, "f_*"] \\ C_*(Y') \arrow[r, hook, "i'_*"] & C_*(X') \arrow[r, two heads, "\mathrm{coker}"] & C_*(X',Y') \end{tikzcd}$$

$\Rightarrow$ We get functors

$$\begin{tikzcd} \mathrm{Pair} \arrow[r, "C_*"] & \mathrm{Ch}_{\geq 0}(\mathrm{Ab}) \arrow[r, "H_n"] \arrow[loop below, "-\otimes A"] & \mathrm{Ab} \end{tikzcd}$$

Change of coefficients: $C_*(X,A)$ and $H_*(X,A)$ are also functorial in A.

Module structures: If R is a ring and M is a left R-module, then $C_*(X) \otimes M$ is a chain complex of left R-modules

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "C_*(-{,}M)"] \arrow[dr, "C_*(-{,}M)"'] & \mathrm{Ch}_{\geq 0}(\mathrm{Mod}_R) \arrow[r, "H_n"] \arrow[d, "\mathrm{forget}"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Mod}_R \arrow[d, "\mathrm{forget}"] \\ & \mathrm{Ch}_{\geq 0}(\mathrm{Ab}) \arrow[r, "H_n"] & \mathrm{Ab} \end{tikzcd}$$

Theorem ("the Eilenberg–Steenrod axioms")

The functors $H_n: \mathrm{Pair} \to \mathrm{Ab}$, $n \in \mathbb{Z}$, satisfy the following properties:

1) (Homotopy invariance) If $f, g: (X,Z) \to (Y,W)$ are homotopic (i.e.: $\exists$ a homotopy $H: X \times I \to Y$ from f to g such that $H(Z \times I) \subset W$), then

$$f_* = g_*: H_n(X,Z) \to H_n(Y,W)$$

2) (Long exact sequence of a pair) For any pair (X,Y), there is a natural long exact sequence

$$\cdots \to H_n(Y) \to H_n(X) \to H_n(X,Y) \xrightarrow{\partial} H_{n-1}(Y) \to \cdots$$

3) (Additivity) $$H_n\Big(\coprod_{i \in I} X_i\Big) \xleftarrow{\cong} \bigoplus_{i \in I} H_n(X_i)$$

4) (Excision) Let $V \subset U \subset X$ with $\overline{V} \subset U^\circ$. Then

$$H_n(X \setminus V, U \setminus V) \xrightarrow{\cong} H_n(X, U).$$

5) (Dimension)

$$H_n(*) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

Remark. A collection of functors $h_n : \mathrm{Pair} \to \mathrm{Ab}$, $n \in \mathbb{Z}$, with natural transformations $\partial : h_n(X,Y) \to h_{n-1}(Y)$ satisfying axioms 1)–4) is called a generalized homology theory. It is called an ordinary homology theory if $h_*(*)$ is concentrated in degree 0. It turns out that any ordinary homology theory (h_*, ∂) coincides with $H_*(-, h_0(*))$ on all "nice enough" spaces (e.g. spaces homotopy equivalent to $|X|$ for some $X \in \mathrm{sSet}$, which includes topological manifolds).
So other generalized homology theories have different values on $*$:

e.g.:
- K-theory: $K_*(*) \cong \mathbb{Z}[\beta^{\pm 1}]$, $\beta^i \in K_{2i}(*)$
- complex bordism: $\Omega^U_*(*) \cong \mathbb{Z}[x_1, x_2, \dots]$, $x_i \in \Omega^U_{2i}(*)$.

The "easy" axioms

Prop (Dimension axiom) $H_n(*, A) = \begin{cases} A & \text{if } n=0 \\ 0 & \text{if } n \neq 0. \end{cases}$

Proof. $\mathrm{Sing}(*) = *$ has a unique non-degenerate simplex.

$$N_*(\mathbb{Z}[\mathrm{Sing}(*)] \otimes A) : \quad \cdots \to \overset{1}{0} \to \overset{0}{A} \to \overset{-1}{0} \to \cdots$$ $\square$

Prop (Generalized additivity)
Let $X \in \mathrm{Top}$, $Y \subset X$, let $(X_i)_{i \in I}$ be the path-connected components of X. Then

$$H_n(X, Y, A) \cong \bigoplus_{i \in I} H_n(X_i, Y \cap X_i, A)$$

Pf. Δ^n is path-connected $\Rightarrow$ $\mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X) \xleftarrow{\cong} \coprod_{i \in I} \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X_i)$

$$\Rightarrow \mathrm{Sing}(X) \cong \coprod_{i \in I} \mathrm{Sing}(X_i)$$

$$\Rightarrow \begin{array}{ccc} C_*(X) & \cong & \bigoplus_{i \in I} C_*(X_i) \\ \uparrow & & \uparrow \\ C_*(Y) & \cong & \bigoplus_{i \in I} C_*(Y \cap X_i) \end{array} \quad \Rightarrow \text{the same for } C_*(X,Y) \text{ and } C_*(X,Y) \otimes A.$$

Finally, $H_n : \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Ab}$ preserves coproducts (easy). $\square$