Prop $H_0(X,A) \cong \bigoplus_{\pi_0(X)} A$

path-connected comp

Pf. Exercise 7.1: $H_0(\mathrm{Sing}\, X, A) \cong \bigoplus_{\pi_0 \mathrm{Sing}(X)} A$ and $\pi_0 \mathrm{Sing}(X) = \pi_0(X)$ $\square$.

Prop (Long exact sequence of a triple) Let $X \in \mathrm{Top}$, $Z \subset Y \subset X$, $A \in \mathrm{Ab}$.
Then there is a long exact sequence, natural in (X,Y,Z) and A,

$$\cdots \to H_n(Y,Z,A) \to H_n(X,Z,A) \to H_n(X,Y,A) \xrightarrow{\partial} H_{n-1}(Y,Z,A) \to \cdots$$

Proof. This is the LES associated with the short exact sequence of chain complexes:

$$0 \to C_*(Y)/C_*(Z) \otimes A \to C_*(X)/C_*(Z) \otimes A \to C_*(X)/C_*(Y) \otimes A \to 0.$$

$\square$.

Proposition (Homotopy invariance)

1) if $f,g : (X,Z) \to (Y,W)$ are homotopic via $H : X \times I \to Y$ such that $H(Z \times I) \subset W$, then $f_* = g_* : H_*(X,Z,A) \to H_*(Y,W,A)$.

2) Let $f : (X,Z) \to (Y,W)$. If both $f : X \to Y$ and $f|_Z : Z \to W$ are homotopy equivalences, then $f_* : H_n(X,Z,A) \xrightarrow{\cong} H_n(Y,W,A)$.

Proof. 1) $H : X \times I \to Y$ homotopy from f to g

$H|_{Z\times I} : Z \times I \to W$ homotopy from $f|_Z$ to $g|_Z$.

$$\begin{tikzcd}
 & & \mathrm{Sing}(X) \arrow[lld, bend right, "id \times \delta_1"'] \arrow[d, "\mathrm{Sing}(i_0)"] \arrow[rd, bend left, "\mathrm{Sing}(f)"] \arrow[ld, phantom, "\circlearrowleft"] & \\
\mathrm{Sing}(X) \times \Delta[1] \arrow[r, "id \times \iota"] & \mathrm{Sing}(X) \times \mathrm{Sing}(I) \cong \mathrm{Sing}(X \times I) \arrow[r, phantom, ""] & \mathrm{Sing}(X \times I) \arrow[r, "\mathrm{Sing}(H)"] & \mathrm{Sing}(Y) \\
 & & \mathrm{Sing}(X) \arrow[llu, bend left, "id \times \delta_0"] \arrow[u, "\mathrm{Sing}(i_1)"'] \arrow[ru, bend right, "\mathrm{Sing}(g)"'] \arrow[lu, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

There is a canonical map $\iota : \Delta[1] \to \mathrm{Sing}(\Delta^1)$ corresponding to id_{Δ^1}, s.t.
$\iota \circ \delta_0 = \mathrm{Sing}(i_1)$
$\iota \circ \delta_1 = \mathrm{Sing}(i_0)$

$\delta_{0,1} : \Delta[0] \to \Delta[1]$

$\Rightarrow \tilde{H} = \mathrm{Sing}(H) \circ (id \times \iota)$ is a simplicial homotopy from $\mathrm{Sing}(g)$ to $\mathrm{Sing}(f)$, which restricts to a simplicial homotopy from $\mathrm{Sing}(g|_Z)$ to $\mathrm{Sing}(f|_Z)$

$$\rightsquigarrow \quad \begin{tikzcd} \mathrm{Sing}(X)_n \arrow[r, "h_{n,i} = \tilde{H}(s_i, \theta_{n+1,i})"] & \mathrm{Sing}(Y)_{n+1} \\ \mathrm{Sing}(Z)_n \arrow[u, hook] \arrow[r, dashed, "h_{n,i}"] & \mathrm{Sing}(W)_{n+1} \arrow[u, hook] \end{tikzcd} \qquad \text{semi-simplicial homotopy}$$

$$\rightsquigarrow \quad \begin{tikzcd} C_n(X) \arrow[r, "h_n = \sum_{i=0}^{n} (-1)^i h_{n,i}"] & C_{n+1}(Y). \\ C_n(Z) \arrow[u, hook] \arrow[r, dashed, "h_n"] & C_{n+1}(W). \arrow[u, hook] \end{tikzcd} \qquad \text{chain-homotopy}$$

$\rightsquigarrow$ induced chain homotopy $\bar{h}_n : C_n(X,Z)\otimes A \longrightarrow C_{n+1}(Y,W)\otimes A$

$\Longrightarrow f_* = g_*$ on homology.

2) $f : X \to Y$ homotopy equivalence $\Rightarrow f_* : H_*(X,A) \xrightarrow{\cong} H_*(Y,A)$ by 1).

$$\begin{tikzcd} \cdots \arrow[r] & H_n(Z) \arrow[r] \arrow[d, "\cong"', "(f|_Z)_*"] & H_n(X) \arrow[r] \arrow[d, "\cong"', "f_*"] & H_n(X,Z) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(Z) \arrow[r] \arrow[d, "\cong"'] & \cdots \arrow[d, "\cong"] \\ \cdots \arrow[r] & H_n(W) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,W) \arrow[r, "\partial"] & H_{n-1}(W) \arrow[r] & \cdots \end{tikzcd}$$

$\Rightarrow f_* : H_n(X,Z) \to H_n(Y,W)$ is an isomorphism by the 5-lemma. $\square$

Consequences of the axioms.

Reduced homology $\tilde{H}_*(X)$ is the homology of the complex

$$\begin{array}{ccccccccc} & 2 & & 1 & & 0 & & -1 & \\ \cdots \to & C_2(X) & \to & C_1(X) & \to & C_0(X) & \xrightarrow{\varepsilon} & \mathbb{Z} & \to 0 \to \cdots \end{array}$$

$$\varepsilon\left(\sum n_i x_i\right) = \sum n_i$$

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & \text{if } n \geq 1 \\ \ker\left(H_0(X) \xrightarrow{\varepsilon} \mathbb{Z}\right) & \text{if } n = 0 \\ \mathrm{coker}\left(H_0(X) \xrightarrow{\varepsilon} \mathbb{Z}\right) & \text{if } n = -1, \; = \begin{cases} 0 & \text{if } X \neq \emptyset \\ \mathbb{Z} & \text{if } X = \emptyset. \end{cases} \\ 0 & \text{if } n \leq -2 \end{cases}$$

If $x_0 \in X$, then $\mathbb{Z} = C_0(\{x_0\}) \overset{\varepsilon}{\underset{}{\leftrightarrows}} C_0(X) \longrightarrow C_0(X, \{x_0\})$

$$\Longrightarrow \quad \tilde{H}_*(X) \cong H_*(X, \{x_0\}).$$

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}.$$

Prop. (NDR pairs)

Let (X,Y) be a pair such that:

- Y is closed in X
- Y is a neighborhood deformation retract (NDR):

  $\exists$ a neighborhood V of Y in X and a map $r: V \to Y$ such that $r|_Y = id_Y$ and $\exists$ a homotopy rel Y $V \times I \to V$ from id_V to r.

Then $H_*(X,Y) \cong \tilde{H}_*(X/Y)$.

Proof. Let $q: X \to X/Y$ be the quotient map.

$$\begin{tikzcd}
H_n(X,Y) \arrow[r, "\cong"', "Y \hookrightarrow V \text{ is htpy equiv.}"] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "q_*"] & H_n(X \setminus Y, V \setminus Y) \arrow[l, "\cong", "\text{excision}"'] \arrow[d, "\cong", "(q|_{X \setminus Y})_*"'] \\
\tilde{H}_n(X/Y) \cong H_n(X/Y, Y/Y) \arrow[r, "\cong", "Y/Y \hookrightarrow V/Y \text{ is a htpy equiv.}"'] & H_n(X/Y, V/Y) & H_n(X/Y \setminus Y/Y, V/Y \setminus Y/Y). \arrow[l, "\cong", "\text{excision}"']
\end{tikzcd}$$

($q|_{X \setminus Y}$ is homeo) $\square$

Example (Homology of spheres) $n \geq 1$.

S^{n-1} is a closed NDR in D^n (e.g. take $V = \{x \in D^n \mid \|x\| > \frac{1}{2}\}$)

$\Rightarrow H_*(D^n, S^{n-1}) \cong \tilde{H}_*(D^n/S^{n-1}) \cong \tilde{H}_*(S^n)$.

LES of this pair:

$$\cdots \to H_k(S^{n-1}) \to H_k(D^n) \to \tilde{H}_k(S^n) \xrightarrow{\partial} H_{k-1}(S^{n-1}) \to H_{k-1}(D^n) \to \cdots$$

$D^n \simeq * \Rightarrow H_k(D^n) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow H_k(S^n) \cong H_{k-1}(S^{n-1})$ for $k \geq 2$.

and

$$0 \to H_1(S^n) \xrightarrow{\partial} \underbrace{H_0(S^{n-1})}_{\begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } n=1 \\ \mathbb{Z} & \text{if } n \geq 2. \end{cases}} \to \mathbb{Z} \to \underset{\substack{\| \\ 0 \text{ because } S^n \text{ path-connected}}}{\tilde{H}_0(S^n)} \to 0$$

$$H_k(S^0) \overset{\text{additivity + dimension}}{=} \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

By induction on n: $$H_k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k=0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases} \qquad (n \geq 1)$$

<u>Prop</u> (Mayer–Vietoris sequence)

$X \in \mathrm{Top}$, $U, V \subset X$ subspaces such that $X = U^\circ \cup V^\circ$, $Y \subset U \cap V$.

Let $j_U : U \hookrightarrow X$, $j_V : V \hookrightarrow X$, $i_U : U \cap V \hookrightarrow U$, $i_V : U \cap V \hookrightarrow V$.

Then there is a long exact sequence

$$\cdots \to H_n(U \cap V, Y, A) \xrightarrow{(i_{U*}, -i_{V*})} H_n(U, Y, A) \oplus H_n(V, Y, A) \xrightarrow{(j_{U*}, j_{V*})} H_n(X, Y, A) \xrightarrow{\partial} H_{n-1}(U \cap V, Y, A) \to \cdots$$

<u>Proof</u>. LES of triples (X, U, Y) and $(V, U \cap V, Y)$:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(U,Y) \arrow[r, "j_{U*}"] & H_n(X,Y) \arrow[r, "id_*"] & H_n(X,U) \arrow[r, "\partial"] & H_{n-1}(U,Y) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_n(U \cap V, Y) \arrow[u, "i_{U*}"] \arrow[r, "i_{V*}"'] & H_n(V,Y) \arrow[u, "j_{V*}"] \arrow[r, "id_*"'] & H_n(V, U \cap V) \arrow[u, "\cong"', "j_{V*}"] \arrow[r, "\partial"] & H_{n-1}(U \cap V, Y) \arrow[u, "i_{U*}"] \arrow[r] & \cdots
\end{tikzcd}$$

excision $\cong$ ($\overline{X \setminus V} \subset U^\circ$)

$\overset{\text{Ex. 8.1}}{\Longrightarrow}$ we get the desired LES. $\square$