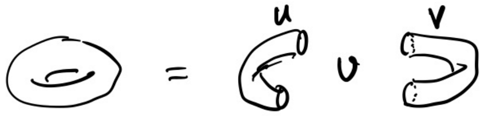
Example $T = S^1 \times S^1$



$U \cap V \simeq S^1 \sqcup S^1$

$$\begin{tikzcd} & S^1 \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] \arrow[dd, dotted, "\text{reflection}"] & \\ U \arrow[r, phantom, "\circlearrowleft"] & & V \arrow[l, phantom, "\circlearrowleft"] \\ & S^1 \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

commutative in hTop

$\Rightarrow$ can choose isomorphisms $H_1(-) \cong \mathbb{Z}$ such that all maps induce $\mathrm{id}_{\mathbb{Z}}$.

$$\cdots \to \cancel{H_2(U) \oplus H_2(V)} \,(=0) \to H_2(T) \xrightarrow{\partial} H_1(U\cap V) \xrightarrow{(i_{U*}, -i_{V*})} H_1(U) \oplus H_1(V) \xrightarrow{(j_{U*}, j_{V*})} H_1(T) \xrightarrow{\partial} H_0(U \cap V) \xrightarrow{(i_{U*}, -i_{V*})} H_0(U)\oplus H_0(V) \to H_0(T) \cong \mathbb{Z} \to 0$$

$H_2(T) \cong \ker = \mathbb{Z}$, with $\ker \hookrightarrow H_1(U\cap V)$

$H_1(U\cap V) \cong \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{array}{c} i_{U*} \\ -i_{V*} \end{array}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}\oplus\mathbb{Z} \cong H_1(U)\oplus H_1(V)$

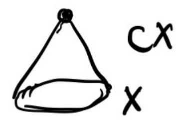
coker $\to \mathbb{Z} \hookrightarrow H_1(T)$, $\quad H_1(T) \to \mathbb{Z} \hookrightarrow H_0(U\cap V)$ ker

$H_0(U\cap V) \cong \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\begin{array}{c} i_{U*} \\ -i_{V*} \end{array}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}\oplus\mathbb{Z} \cong H_0(U)\oplus H_0(V)$

$$\Rightarrow \quad H_n(T) \cong \begin{cases} 0 & \text{if } n \geq 3 \\ \mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 0 \end{cases}$$

Mapping cones and suspensions

Def. Let $f: Y \to X$. The <u>cone</u> on X is $CX = X \times I / X \times \{1\}$



The <u>mapping cone</u> Cf is the pushout

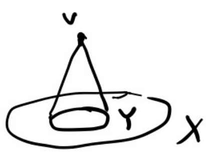
$$\begin{tikzcd} y \arrow[d, mapsto] & Y \arrow[r, "f"] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{PO}"] & X \arrow[d] \\ (y,0) & CY \arrow[r] & Cf \end{tikzcd}$$

Note: $CX \simeq *$

Prop: For any pair (X,Y), $H_*(X,Y) \cong \tilde{H}_*(Ci)$ where $i: Y \hookrightarrow X$.

Pf.

Ci:



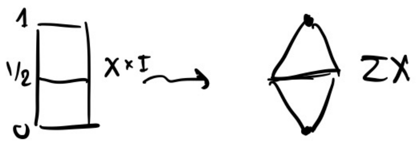
Let $v \in CY$ be the vertex of the cone.

$Y \hookrightarrow CY \setminus \{v\}$
$X \hookrightarrow Ci \setminus \{v\}$ are homotopy equivalences

$$\begin{tikzcd} H_*(X,Y) \arrow[r] \arrow[dr, "\cong"', "(HI)"] & H_*(Ci, CY) & H_*(Ci, \{v\}) \arrow[l, "\cong (HI)"'] \cong \tilde{H}_*(Ci). \quad \square \\ & H_*(Ci\setminus\{v\}, CY\setminus\{v\}) \arrow[u, "\cong (\text{excision})"'] & \end{tikzcd}$$

<u>Recall:</u> $\Sigma X = X \times I \coprod_{X\times\{0,1\}} \{0,1\} = C(X \to *)$

$$\Sigma S^n \cong S^{n+1}.$$



<u>Prop</u> (Suspension isomorphism)

There is a natural isomorphism $\tilde{H}_n(\Sigma X) \underset{\sigma}{\cong} \tilde{H}_{n-1}(X)$ for all $n \in \mathbb{Z}$.

<u>Pf.</u>

$$C_+X = X \times (\tfrac{1}{4}, 1] \,/\, X \times \{1\}$$

$$C_-X = X \times [0, 3/4) \,/\, X \times \{0\}$$

$$\Sigma X = C_+X \cup C_-X, \quad C_+X \simeq * \simeq C_-X, \quad C_+X \cap C_-X \cong X \times (\tfrac{1}{4}, \tfrac{3}{4}) \simeq X.$$

Mayer-Vietoris + homotopy invariance:

$$\cdots \to H_n(X) \to \overset{0}{\cancel{H_n(*) \oplus H_n(*)}} \to H_n(\Sigma X) \xrightarrow{\partial} H_{n-1}(X) \to \overset{0}{\cancel{H_{n-1}(*) \oplus H_{n-1}(*)}}$$

$$0 \to H_1(\Sigma X) \xrightarrow{\partial} H_0(X) \xrightarrow{(\varepsilon, -\varepsilon)} H_0(*) \oplus H_0(*) \to H_0(\Sigma X) \to 0$$

$$\Rightarrow \quad H_n(\Sigma X) \cong \begin{cases} H_{n-1}(X) & \text{if } n \geq 2 \\ \ker(\varepsilon) = \tilde{H}_0(X) & \text{if } n = 1 \\ \operatorname{coker}(\varepsilon, -\varepsilon) = \begin{cases} \mathbb{Z} & \text{if } X \neq \emptyset \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } X = \emptyset \end{cases} & \text{if } n = 0. \end{cases}$$

$$\Rightarrow \quad \tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X).$$

$\square$

<u>Remark.</u> This gives another way of computing $H_*(S^n)$, since $S^n = \Sigma S^{n-1}$ and $S^0 = \Sigma \emptyset$.

<u>Mapping degree.</u>

<u>Definition</u> Let $n \geq 1$ and $f: S^n \to S^n$ a continuous map. The <u>degree</u> of f is the unique integer $\deg(f) \in \mathbb{Z}$ such that:

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \cong & & \cong \\ \mathbb{Z} & \dashrightarrow & \mathbb{Z} \\ x & \longmapsto & \deg(f) \cdot x \end{array}$$

Rmk. More generally, if M is a compact connected orientable smooth manifold of dim n, then $H_n(M,\mathbb{Z}) \cong \mathbb{Z}$, so we can define $\deg(f)$ for any $f: M \to M$.

Proposition Let $n \geq 1$.

1) $\deg: [S^n, S^n] \longrightarrow \mathbb{Z}$ is surjective.
2) $\deg(g \circ f) = \deg(g)\deg(f)$
3) If $f: S^n \to S^n$ is null-homotopic, then $\deg(f) = 0$
4) $\deg(\Sigma f) = \deg(f)$ where $\Sigma f: S^{n+1} \to S^{n+1}$.
5) $\deg(S^n \to S^n, x \mapsto -x) = (-1)^{n+1}$
6) if $r: S^n \to S^n$ is a reflection (across a hyperplane through the origin) then $\deg(r) = -1$.
7) $\deg(S^1 \to S^1, z \mapsto z^d) = d$ for any $d \in \mathbb{Z}$.

Proof. 1) follows from 7) & 4).

2) $H_n(g \circ f) = H_n(g) \circ H_n(f)$
$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\cdot n"'] \arrow[rr, bend left, "\cdot(nm)"] & \mathbb{Z} \arrow[r, "\cdot m"'] & \mathbb{Z} \end{tikzcd}.$$

3) $f \simeq \text{const} \implies f_* = \text{const}_*: H_n(S^n) \longrightarrow H_n(S^n)$

$$\begin{tikzcd} S^n \arrow[rr, "\text{const}"] \arrow[dr] & & S^n \\ & * \arrow[ur] & \end{tikzcd}$$
$H_n(*) = 0 \implies \text{const}_* = 0$

4)
$$\begin{tikzcd} H_n(S^n) \arrow[r, "\cong", "\sigma"'] \arrow[d, "f_*"'] & H_{n+1}(S^{n+1}) \arrow[d, "(\Sigma f)_*"] \\ H_n(S^n) \arrow[r, "\cong", "\sigma"'] & H_{n+1}(S^{n+1}) \end{tikzcd}$$
commutes by naturality of σ.

5) $x \mapsto -x$ is the composition of $n+1$ reflections
$\implies$ 5) follows from 6) & 2).

6) If r, r' are reflections, $\exists\, \varphi: S^n \xrightarrow{\cong} S^n$ s.t. $r = \varphi^{-1} \circ r' \circ \varphi$
$\implies \deg(r) = \deg(r')$.

Exercise 8: $\exists$ pinch $: S^n \longrightarrow S^n \vee S^n$ s.t. $p_i \circ \text{pinch} \simeq \mathrm{id}_{S^n}$

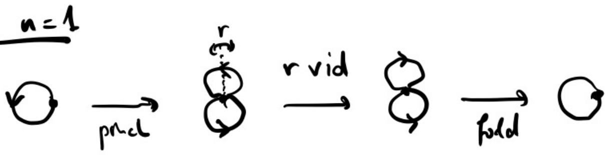
$$\begin{tikzcd} S^n \vee S^n \arrow[d, "p_1, p_2"] \\ S^n \end{tikzcd}$$

$f, g : S^n \longrightarrow S^n$

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\Delta"] \arrow[dr, "\mathrm{pinch}_*"'] & H_n(S^n) \oplus H_n(S^n) \arrow[r, "f_* \oplus g_*"] & H_n(S^n) \oplus H_n(S^n) \arrow[r, "+"] \arrow[d, "(i_{1*}, i_{2*})", "\cong"'] & H_n(S^n) \\
& H_n(S^n \vee S^n) \arrow[u, "(p_{1*}, p_{2*})", "\cong"'] \arrow[r, "(f \vee g)_*"] & H_n(S^n \vee S^n) \arrow[ur, "\mathrm{fold}_*"'] &
\end{tikzcd}$$

$\Longrightarrow \quad f_* + g_* = \left(\mathrm{fold} \circ (f \vee g) \circ \mathrm{pinch}\right)_* \qquad (*)$

Case $n=1$



is null-homotopic

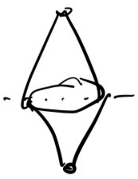
i.e. is zero $[S^1, S^1]_* = \pi_1(S^1)$

indeed, it is $[\overline{id_{S^1}} * id_{S^1}]_{p.} = 0$.

$\Longrightarrow \quad r_* + id_* = 0 \quad$ by $(*)$

$\Longrightarrow \quad \deg(r) = -\deg(id) = -1$.

Case $n \geqslant 2$ induction on n



$r : S^{n-1} \longrightarrow S^{n-1}$ reflection $\Longrightarrow \Sigma r : S^n \longrightarrow S^n$ is also a reflection

by 4), $\deg(\Sigma r) = \deg(r) = -1$.

7). **Case $d=0$.** $z \mapsto z^0$ is constant $\Longrightarrow \deg(z \mapsto z^0) = 0$ by 3)

Case $d>0$

is d in $\pi_1(S^1) \cong \mathbb{Z}$

$\Longrightarrow \quad z^d \simeq \mathrm{fold} \circ (z^{d-1} \vee id) \circ \mathrm{pinch}$

$\Longrightarrow \quad (z^d)_* = \left(\mathrm{fold} \circ (z^{d-1} \vee id) \circ \mathrm{pinch}\right)_* \overset{\text{by } (*)}{=} \underbrace{(z^{d-1})_*}_{\substack{\text{mult. by} \\ d-1 \\ \text{by induction}}} + \underbrace{id_*}_{id}$

$\Longrightarrow \quad (z^d)_* = \text{mult. by } d$.

Case $d<0$: $\quad (z \mapsto z^{-d}) = r \circ (z \mapsto z^d)$

by 2) & 6), $\deg(z^{-d}) = -\deg(z^d)$. $\square$.