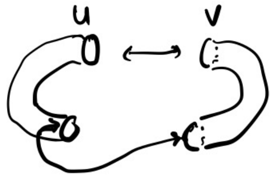
Remark. If $f, g : S^n \to S^n$ have the same degree, then $f \simeq g$.

This follows from the computation $\pi_n(S^n) \xrightarrow[\deg]{\cong} \mathbb{Z}$ (this is non-trivial).

Example. K = Klein bottle



$$\begin{tikzcd}
 & S^1 \arrow[dl, hook] \arrow[dr, hook] \arrow[dd, "\cong"'] & \\
U \arrow[r, phantom, "\circlearrowleft \simeq"] & & V \arrow[l, phantom, "\circlearrowleft \simeq"] \\
 & S^1 \arrow[ul, hook] \arrow[ur, hook] \arrow[loop right, "r"] &
\end{tikzcd}$$

$U \simeq S^1$, $V \simeq S^1$, $U \cap V \simeq S^1 \amalg S^1$.

Meyer-Vietoris:

$$\begin{tikzcd}
\overset{0}{\cancel{H_2(U) \oplus H_2(V)}} \arrow[r] & H_2(K) \arrow[r, "\partial"] \arrow[dd, "\cong"'] & H_1(U \cap V) \arrow[r, "{i_{U*}, -i_{V*}}"] & H_1(U) \oplus H_1(V) \arrow[r] & H_1(K) \arrow[r, "\partial"] \arrow[ddr] & H_0(U \cap V) \arrow[r, "{i_{U*}, -i_{V*}}"] \arrow[d, phantom, "\cong"] & H_0(U) \oplus H_0(V) \arrow[d, phantom, "\cong"] \\
 & & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\begin{array}{c|cc} i_{U*} & 1 & 1 \\ -i_{V*} & -1 & 1 \end{array}"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[dr, two heads] & & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, phantom, "\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}"] & \mathbb{Z} \oplus \mathbb{Z} \\
 & \ker \arrow[ur, hook] & & & \mathrm{coker} \arrow[uu, hook] & \mathbb{Z} \arrow[uu, hook] &
\end{tikzcd}$$

⚠ $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is not invertible with $\mathbb{Z}$-coefficients. ($\det = 2$)

- injective
- image is $\{(x+y, -x+y)\} \subset \mathbb{Z} \oplus \mathbb{Z}$

$$\begin{aligned} &\{(x+y, -x+y)\} \\ &\quad \parallel \\ &\{(z,w) \mid z-w \text{ is even}\} \end{aligned}$$ subgroup of index 2

$\Rightarrow$ cokernel is $\mathbb{Z}/2$.

$$\Rightarrow \quad H_n(K) = \begin{cases} 0 & \text{if } n \geq 2 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 0. \end{cases}$$

Application: Hedgehog theorem

Definition The *tangent bundle* of S^n is $TS^n = \{(x,v) \in S^n \times \mathbb{R}^{n+1} \mid v \text{ is tangent to } S^n \text{ at } x, \text{ i.e.: } \langle x, v\rangle = 0\}$

A *vector field* is a section of the map $\pi : TS^n \to S^n$, $\pi(x,v) = x$,

i.e.: a map $S^n \to TS^n$ of the form $x \mapsto (x, v(x))$.

Theorem Let $n \geq 1$. Then there exists a non-vanishing continuous vector field on S^n if and only if n is odd.

Proof. Suppose n odd. Set

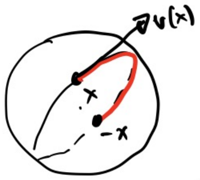
$$v(x) = (x_2, -x_1, x_4, -x_3, \ldots, x_{n+1}, -x_n)$$

Then $\langle x, v(x) \rangle = 0$ so v is a non-vanishing vector field.

Conversely, suppose v is a non-vanishing continuous vector field on S^n:

$$v : S^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\} \quad \text{s.t} \quad \langle x, v(x) \rangle = 0 \text{ for all } x \in S^n.$$

Idea: $H(x,-)$ is the geodesic from x to $-x$ in the direction of $v(x)$.



$\Rightarrow$ H is a homotopy from id_{S^n} to $-\mathrm{id}_{S^n}$.

formula: $H(x,t) = \cos(\pi t) \cdot x + \sin(\pi t) \cdot \dfrac{v(x)}{\|v(x)\|}$.

$$\Rightarrow \quad 1 = \deg(\mathrm{id}_{S^n}) = \deg(-\mathrm{id}_{S^n}) = (-1)^{n+1} \Rightarrow n \text{ odd.} \quad \square$$

Application: Jordan curve theorem

Lemma: Let $X \in \mathrm{Top}$ and $(X_i)_{i \in I}$ a filtered family of subspace of X such that $X = \bigcup_{i \in I} X_i^\circ$. Then $H_*(X) \cong \operatorname{colim}_{i \in I} H_*(X_i)$.

Pf. Δ^n compact $\Longrightarrow$ $\mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X) = \bigcup_{i \in I} \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X_i)$

$\Longrightarrow$ $\mathrm{Sing}(X) = \bigcup_{i \in I} \mathrm{Sing}(X_i)$

$\Longrightarrow$ $C_*(X) = \bigcup_{i \in I} C_*(X_i)$

$\Longrightarrow$ $H_*(X) = \operatorname{colim}_{i \in I} H_*(X_i)$ $\square$.

$H_n : \mathrm{Ch}(\mathrm{Ab}) \longrightarrow \mathrm{Ab}$ preserves filtered colimits.

Theorem Let $n \geq 1$.

1) If $D \subset \mathbb{R}^n$ is homeomorphic to D^n, then $\mathbb{R}^n \setminus D$ is path-connected
2) If $S \subset \mathbb{R}^n$ is homeomorphic to S^{n-1} then $\mathbb{R}^n - S$ has exactly 2 path-connected components.

Remark: There are examples where $\mathbb{R}^n \setminus D$ is not homeomorphic to $\mathbb{R}^n \setminus D^n$. The **Alexander horned ball** is some $D \subset \mathbb{R}^3$ with $D \cong D^3$ such that $\mathbb{R}^3 \setminus D$ is not simply connected.

Pf. Since D & S are compact, they are bounded $\Rightarrow$ we can replace $\mathbb{R}^n$ by its one-pt compactification $\mathbb{R}^n \cup \{\infty\} \cong S^n$.

Recall that
$$H_0(X) \cong \bigoplus_{\pi_0(X)} \mathbb{Z} \cong \mathbb{Z} \oplus \tilde{H}_0(X).$$

So the theorem follows from:

Theorem $m, n \geq 0$

1) if $f : I^m \longrightarrow S^n$ injective continuous, then
$$\tilde{H}_*\left(S^n \setminus f(I^m)\right) = 0$$
2) if $f : S^m \longrightarrow S^n$ is injective continuous, then
$$\tilde{H}_k\left(S^n \setminus f(S^m)\right) = \begin{cases} \mathbb{Z} & \text{if } k = n-m-1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

1) Induction on m. If $m = 0$, $S^n - f(I^0) \cong \mathbb{R}^n \Rightarrow \tilde{H}_* = 0$.

Write $I^m = I^{m-1} \times [0, \tfrac{1}{2}] \cup I^{m-1} \times [\tfrac{1}{2}, 1]$.

$$U = S^n \setminus f\left(I^{m-1} \times [0, \tfrac{1}{2}]\right)$$
$$V = S^n \setminus f\left(I^{m-1} \times [\tfrac{1}{2}, 1]\right)$$
$$U \cap V = S^n \setminus f(I^m)$$
$$U \cup V \underset{\substack{\uparrow \\ f \text{ injective}}}{=} S^n \setminus f\left(I^{m-1} \times \{\tfrac{1}{2}\}\right)$$

By induction, $\tilde{H}_*(U \cup V) = 0$.

$\Rightarrow$ Mayer–Vietoris: $\tilde{H}_*(U\cap V) \xrightarrow[(i_{U*},\,-i_{V*})]{\cong} \tilde{H}_*(U) \oplus \tilde{H}_*(V)$

Suppose $\alpha \in \tilde{H}_k(U \cap V)$ nonzero. $\Rightarrow$ $i_{U*}(\alpha) \neq 0$
or $i_{V*}(\alpha) \neq 0$

Say $i_{U*}(\alpha) \neq 0$. Write $I^{m-1} \times [0,\frac{1}{2}] = I^{m-1} \times [0,\frac{1}{4}] \cup I^{m-1} \times [\frac{1}{4},\frac{1}{2}]$.

$\rightsquigarrow$ get a sequence of sub-intervals $I_0 = [0,1] \supset I_1 = [0,\frac{1}{2}] \supset I_2 \supset I_3 \supset \cdots$

s.t.

$$\begin{tikzcd}
\tilde{H}_k(U\cap V) \arrow[r] & \tilde{H}_k(S^n \setminus f(I^{m-1} \times I_j)) \\
\rotatebox{90}{$\in$} & \\
\alpha \arrow[r, mapsto] & \neq 0
\end{tikzcd}$$

for all j. (*)

I compact $\Rightarrow$ $\bigcap_{j \geq 0} I_j = \{t\}$

$$\bigcup_{j \geq 0} (S^n \setminus f(I^{m-1} \times I_j)) = S^n \setminus f(I^{m-1} \times \{t\})$$

$$\underset{\text{Lemma}}{\Longrightarrow} \quad \underset{\substack{\| \text{ by induction} \\ 0}}{\tilde{H}_k(S^n \setminus f(I^{m-1} \times \{t\}))} = \operatorname*{colim}_{j \geq 0} \tilde{H}_k(S^n \setminus f(I^{m-1} \times I_j))$$

so $\alpha \mapsto 0$ in $\operatorname*{colim}_{j\geq 0} \tilde{H}_k(\cdots)$

$\Rightarrow$ $\alpha \mapsto 0$ in $\tilde{H}_k(S^n \setminus f(I^{m-1} \times I_j))$
for some j.

This contradicts (*).