2) Induction on m. $m=0$: $S^n \setminus f(S^0) \simeq S^{n-1}$ ✓

$m \geq 1$. $S^m = D^m_+ \underset{S^{m-1}}{\cup} D^m_-$

Let
$$U = S^n \setminus f(D^m_+)$$
$$V = S^n \setminus f(D^m_-)$$
$$U \cup V = S^n \setminus f(S^{m-1})$$
$$U \cap V = S^n \setminus f(S^m)$$

By 1), $\tilde{H}_*(U) = 0$ and $\tilde{H}_*(V) = 0$.

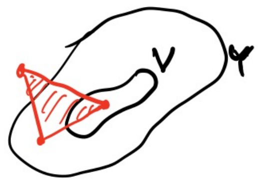
By Mayer-Vietoris: $\tilde{H}_k(S^n \setminus f(S^{m-1})) \underset{\cong}{\xrightarrow{\partial}} \tilde{H}_{k-1}(S^n \setminus f(S^m))$

We conclude by induction. □

Excision

Recall: $V \subset Y \subset X \quad \bar{V} \subset Y^\circ \Rightarrow H_*(X,Y) \xleftarrow{\cong} H_*(X \setminus V, Y \setminus V)$

$$H_*(X,Y) = H_*(C_*(X,Y))$$



$$\begin{tikzcd}
C_*(Y) \arrow[r, hook] & C_*(X) \arrow[r, two heads, "\text{coker}"] & C_*(X,Y) \\
C_*(Y \setminus V) \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\text{not PO !}"] & C_*(X \setminus V) \arrow[u] \arrow[r, two heads, "\text{coker}"] & C_*(X \setminus V, Y \setminus V) \arrow[u]
\end{tikzcd}$$

For $\mathcal{U} = (U_{i \in I})$ a collection of subspaces of X such that $X = \bigcup_{i \in I} U_i^\circ$, we define:

$$\mathrm{Sing}^{\mathcal{U}}(X) = \bigcup_{i \in I} \mathrm{Sing}(U_i) \subset \mathrm{Sing}(X)$$

$$C_*^{\mathcal{U}}(X) = C_*(\mathrm{Sing}^{\mathcal{U}}(X)) \subset C_*(X)$$

(explicitly: $C_n^{\mathcal{U}}(X) \subset C_n(X)$ is the subgroup generated by $C_n(U_i)$ for all $i \in I$.)

Theorem (generalized excision) The inclusion $C_*^{\mathcal{U}}(X) \otimes A \hookrightarrow C_*(X) \otimes A$ is a quasi-isomorphism (for any space X, any abelian group A, any $\mathcal{U}$ as above)

Corollary The excision axiom holds (with any coefficients A)

Pf. $\mathcal{U} = \{Y, X \setminus V\}$. We have SES of chain complexes:

$$\begin{tikzcd}
C_*(Y\setminus V) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{PO}"] & C_*(X\setminus V) \arrow[r, two heads] \arrow[d, hook] & C_*(X\setminus V, Y\setminus V) \arrow[d, "\cong"] \\
C_*(Y) \arrow[r, hook] \arrow[d, equal] & C_*^{\mathcal{U}}(X) \arrow[r, two heads] \arrow[d, "\text{quasi-iso}"] & \operatorname{coker} \arrow[d] \\
C_*(Y) \arrow[r, hook] & C_*(X) \arrow[r, two heads] & C_*(X,Y).
\end{tikzcd}$$

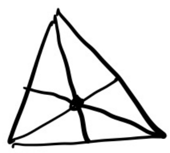
By the 5-lemma, $C_*(X\setminus V, Y\setminus V) \to C_*(X,Y)$ is a quasi-iso. $\square$

Proof of Theorem Let $i : C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$.

We will construct $r : C_*(X) \to C_*^{\mathcal{U}}(X)$ such that both $r \circ i$ and $i \circ r$ are chain-homotopic to the identity.

Since $- \otimes A$ preserves chain homotopies, $i \otimes A$ will be a quasi-isomorphism.

Barycentric subdivision of simplices



Let $v_0, \dots, v_n \in \mathbb{R}^m$

$$[v_0, \dots, v_n] = \left\{ \sum_{i=0}^n t_i v_i \;\middle|\; t_i \geq 0,\ \sum t_i = 1 \right\} \subset \mathbb{R}^m.$$

e.g. $\Delta^n = [e_0, \dots, e_n] \subset \mathbb{R}^{n+1}$

$\exists$ canonical map $\Delta^n \to [v_0, \dots, v_n]$ which is a homeomorphism if the v_i's do not lie on an n-plane.

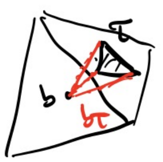
$$\sum t_i e_i \mapsto \sum t_i v_i$$

The **barycenter** of $[v_0, \dots, v_n]$ is $\frac{1}{n+1} \sum_{i=0}^n v_i$.

We define $\sigma_n \in C_n(\Delta^n)$, an alternating sum of subsimplices of the form $[v_0, \dots, v_n] \subset \Delta^n$, by induction on n:

$\underline{n=0}$: $\sigma_0 = (\Delta^0 \to \Delta^0) \in C_0(\Delta^0)$

$\underline{n>0}$: $\delta_i : \Delta^{n-1} \hookrightarrow \Delta^n$, $0 \leq i \leq n$. Let $b =$ barycenter of Δ^n.



For $\tau : \Delta^{n-1} \to \Delta^n$, let

$$\begin{tikzcd}
b\tau : \Delta^n \arrow[r] \arrow[dr, "\text{can}"'] & \Delta^n \\
& {[b, \tau(e_0), \dots, \tau(e_{n-1})]} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$$\sigma_{n-1} \in C_{n-1}(\Delta^{n-1})$$

$$\sigma_n = \sum_{i=0}^{n} (-1)^i \, b\, \delta_{i*}(\sigma_{n-1})$$

We define the <u>subdivision operator</u> $S : C_n(X) \longrightarrow C_n(X)$

$$(\sigma : \Delta^n \to X) \longmapsto \sigma_*(\sigma_n)$$

where $\sigma_* : C_n(\Delta^n) \longrightarrow C_n(X)$

<u>Claim</u>: $d\sigma_n = \sum_{i=0}^{n} (-1)^i \delta_{i*}(\sigma_{n-1})$ in $C_{n-1}(\Delta^n)$.

$$d([v_0, \dots, v_n]) = \sum (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$

$$\Rightarrow \quad d(b\tau) = \tau - b\, d(\tau) \qquad (*)$$

$$\Rightarrow \quad d\sigma_n = \sum_{i=0}^{n} (-1)^i \delta_{i*}(\sigma_{n-1}) - b\Big(\underbrace{\sum (-1)^i \delta_{i*}(d\sigma_{n-1})}\Big)$$

$$\underset{\text{induction}}{=} \sum_i (-1)^i \delta_{i*}\Big(\sum_j (-1)^j \delta_{j*}(\sigma_{n-2})\Big)$$

$$= 0 \quad \text{(cf. } d^2 = 0 \text{ in a simplicial abelian group)}$$

- S is a chain map: $dS = Sd$

Let $\sigma : \Delta^n \to X$ $\qquad d\sigma = \sum (-1)^i d_i \sigma$

$$S(d\sigma) = \sum (-1)^i S(d_i\sigma) = \sum (-1)^i (\sigma \circ \delta_i)_*(\sigma_{n-1}) = \sigma_*\Big(\sum (-1)^i \delta_{i*}(\sigma_{n-1})\Big)$$

$$d\,S(\sigma) = d(\sigma_*(\sigma_n)) = \sigma_*(d\sigma_n) \quad \underset{\text{by the claim}}{=\!=}$$

- S is chain-homotopic to id.

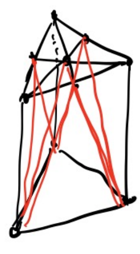
$$T : C_n(X) \longrightarrow C_{n+1}(X) \quad \text{s.t.} \quad dT + Td = \mathrm{id} - S.$$

$$T(\Delta^n \xrightarrow{\sigma} X) = \sigma_*(\tau_n) \quad \text{for some } \tau_n \in C_{n+1}(\Delta^n).$$

Define τ_n by induction: $\tau_0 = (\Delta^1 \to \Delta^0) \in C_1(\Delta^0)$.

$$\tau_n = b\Big(\iota_n - \sum_{i=0}^{n} (-1)^i \delta_{i*}(\tau_{n-1})\Big)$$

where $\iota_n = [e_0, \dots, e_n]$.



$$\begin{array}{ccc} \Delta^n \times I & \longrightarrow & \Delta^n \\ \cup & \nearrow & \\ \Delta^{n+1} & & \end{array}$$

<u>Claim</u> $d\tau_n + \sum (-1)^i \delta_{i*}(\tau_{n-1}) = \iota_n - \sigma_n$ in $C_n(\Delta^n)$

$$d\tau_n \overset{(*)}{=} \iota_n - \sum (-1)^i \delta_{i*}(\tau_{n-1}) - b\Big(\underbrace{d\iota_n - \sum (-1)^i \delta_{i*}(d\tau_{n-1})}_{\underset{\text{induction}}{=} \cdots = \sigma_n}\Big)$$

$\Rightarrow dT + Td = \mathrm{id} - S$ as before.

- Note that both S and T preserve $C^{\mathcal{U}}_*(X) \subset C_*(X)$
- For every $\sigma : \Delta^n \to X$, $\exists\, e \geq 0$ s.t. $S^e(\sigma) \in C^{\mathcal{U}}_n(X)$:
  - if $[v_0, \dots, v_n] \in C_n(\Delta^n)$ is a n-simplex of diameter d then all the simplices in $S([v_0, \dots, v_n])$ have diameter $\leq d \cdot \frac{n}{n+1}$

$$\left[\begin{array}{l} \|v_i - b_j\| \leq \frac{n-1}{n} \|v_i - v_j\| \text{ by induction} \\ \|v_i - b\| = \frac{n}{n+1} \|v_i - b_i\| \leq \frac{n}{n+1} \|v_i - v_j\| \end{array} \right]$$

  - by the Lebesgue lemma applied to Δ^n, $\exists\, \varepsilon > 0$ s.t. for all $x \in \Delta^n$, $\exists\, i \in I$ s.t. $\sigma(B(x, \varepsilon)) \subset \mathcal{U}_i$.
  - hence, if $\sqrt{n}\left(\frac{n}{n+1}\right)^e < 2\varepsilon$, then $S^e(\sigma) \in C^{\mathcal{U}}_n(X)$.

- For any $\sigma : \Delta^n \to X$, choose $e(\sigma)$ s.t. $S^{e(\sigma)}(\sigma) \in C^{\mathcal{U}}_n(X)$.

Let
$$\tilde{T} : C_n(X) \longrightarrow C_{n+1}(X)$$
$$\sigma \longmapsto \sum_{i=0}^{e(\sigma)-1} (T \circ S^i)(\sigma)$$

Define $\tilde{S} = \mathrm{id} - d\tilde{T} - \tilde{T}d : C_n(X) \longrightarrow C_n(X)$

$$d\tilde{S} = d - \cancel{d^2\tilde{T}}^{\,0} - d\tilde{T}d$$
$$\tilde{S}d = d - d\tilde{T}d - \cancel{\tilde{T}d^2}^{\,0}$$

So $\tilde{S}$ is a chain map and $\tilde{T}$ is a chain homotopy between id and $\tilde{S}$.
Also, $\tilde{S}$ & $\tilde{T}$ preserve $C^{\mathcal{U}}_*$ because S & T do.

<u>Claim</u>: $\tilde{S} : C_*(X) \longrightarrow C^{\mathcal{U}}_*(X)$.

Given this claim, we have:

$C_*(X) \xrightarrow{\tilde{S}} C^{\mathcal{U}}_*(X) \hookrightarrow C_*(X)$ is chain-homotopic to id via $\tilde{T}$

$C^{\mathcal{U}}_*(X) \hookrightarrow C_*(X) \xrightarrow{\tilde{S}} C^{\mathcal{U}}_*(X)$ is ______

$$\tilde{S}(\sigma) = \sigma - \sum_{i=0}^{e(\sigma)-1} \left(dTS^i\sigma + TdS^i\sigma \right) \qquad dT + Td = \mathrm{id} - S$$
$$= \sigma - \sum_{i=0}^{e(\sigma)-1} (\mathrm{id} - S)S^i\sigma = \sigma - \left(\sigma - S^{e(\sigma)}\sigma\right) = S^{e(\sigma)}(\sigma) \in C^{\mathcal{U}}_n(X). \qquad \square$$