§ 7. Cellular homology (a method to compute singular homology).

Definition.

- Let (X,Y) be a pair of topological spaces. A relative CW structure on (X,Y) is a sequence of subspaces

$$Y = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \cdots \subset X$$

such that:

1) $X = \operatorname{colim}_{n\to\infty} X^{(n)}$

2) for each $n \geq 0$, $X^{(n)}$ is obtained from $X^{(n-1)}$ by "attaching n-cells": there exist maps $\left(\Phi_n^\alpha : D^n \longrightarrow X^{(n)}\right)_{\alpha\in I_n}$ such that $\Phi_n^\alpha(\partial D^n) \subset X^{(n-1)}$ and the following square is a pushout square:

$(S^{-1} = \emptyset)$

$$\begin{tikzcd}
\coprod_{\alpha\in I_n} S^{n-1} \arrow[r, "(\varphi_n^\alpha)"] \arrow[d, hook] \arrow[dr, phantom, "PO"] & X^{(n-1)} \arrow[d, hook] \\
\coprod_{\alpha\in I_n} D^n \arrow[r, "(\Phi_n^\alpha)"] & X^{(n)}
\end{tikzcd}$$

$\varphi_n^\alpha := \Phi_n^\alpha|_{\partial D^n}$.

- $X^{(n)}$ is called the n-skeleton of (X,Y).
  When given, the maps Φ_n^α are called the characteristic maps
  and φ_n^α are called the attaching maps.

- A CW structure on a space X is a relative CW structure on $(X,\emptyset)$

- A relative CW complex is a pair with a relative CW structure.

- A CW complex is a space with a CW structure

- If (X,Y) and (X',Y') are relative CW complexes, a morphism of pairs $f: (X,Y) \longrightarrow (X',Y')$ is called cellular if $f(X^{(n)}) \subset X'^{(n)}$.

- We have categories: **CWPair** : relative CW complexes and cellular maps
  CW : CW complexes and cellular maps.

Remark. If (X,Y) is a relative CW complex, then:

- $X^{(n)} \setminus X^{(n-1)} = \coprod_{\alpha \in I_n} (D^n)^\circ \quad \Rightarrow \quad \# I_n$ is determined by the CW structure

- $X^{(n)} / X^{(n-1)} \cong \left(\coprod_{\alpha \in I_n} D^n \right) \Big/ \left(\coprod_{\alpha \in I_n} S^{n-1} \right) \cong \bigvee_{\alpha \in I_n} S^n$

The connected components of $X^{(n)} \setminus X^{(n-1)}$ are called the (open) <u>n-cells</u>.

$\Rightarrow$ X is the union of its cells and Y.

Definition. A CW complex X

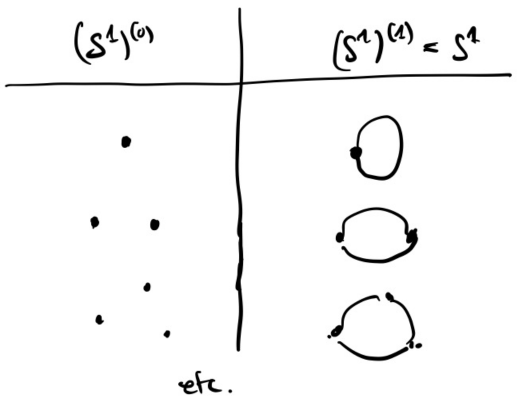
- has <u>dimension n</u> if $X = X^{(n)}$, $X \neq X^{(n-1)}$.
- is <u>finite</u> if it has finitely many cells, i.e. $\coprod_{n \geq 0} I_n$ is finite.

Examples

- Any discrete space has a CW structure of dimension 0.
- (D^n, S^{n-1}) has a relative CW structure with $(D^n)^{(k)} = \begin{cases} S^{n-1} & \text{if } k < n \\ D^n & \text{if } k \geq n \end{cases}$.

$$\begin{tikzcd} S^{n-1} \arrow[r, "id"] \arrow[d, hook] \arrow[dr, phantom, "PO"] & S^{n-1} \arrow[d, hook] \\ D^n \arrow[r, "id"] & D^n \end{tikzcd}$$

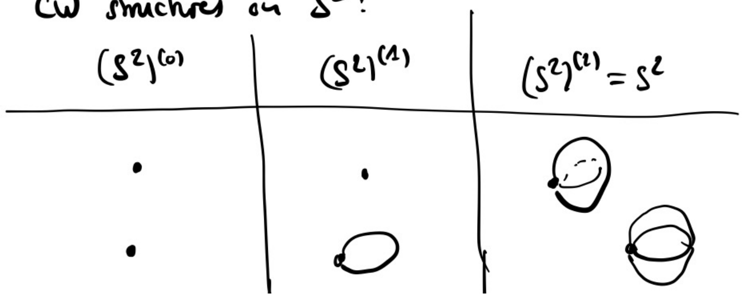
- CW structures on S^1:



1 0-cell, 1 1-cell

2 0-cells, 2 1-cells

- CW structures on S^2:



1 0-cell, 1 2-cell

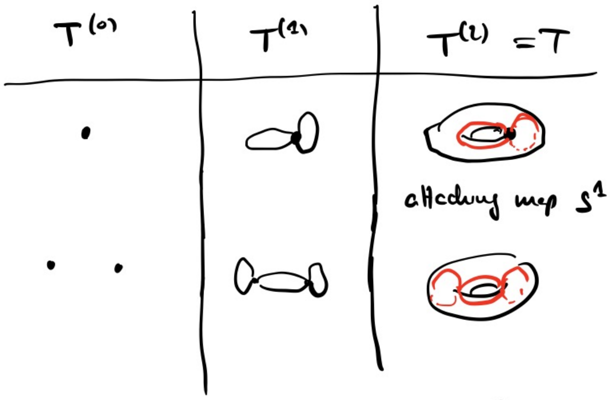
1 0-cell, 1 1-cell, 2 2-cells

2 0-cells, 2 1-cells, 2 2-cells.

- any convex polyhedron defines a CW structure on S^2 with
  0-cells = vertices
  1-cells = edges
  2-cells = faces.

- S^n has a CW structure with $(S^n)^{(k)} = \begin{cases} * & \text{if } k<n \\ S^n & \text{if } k \geq n. \end{cases}$

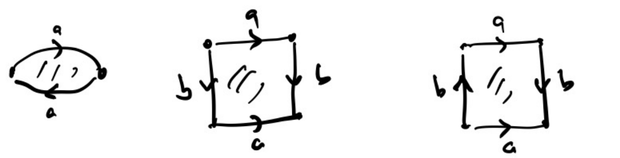
- CW structures on the torus $T = S^1 \times S^1$:



1 0-cell, 2 1-cells, 1 2-cell.
attaching map $S^1 \longrightarrow T^{(1)} = S^1 \vee S^1$ is a pinch map.

2 0-cells, 3 1-cells, 2 2-cells.

- pictures like



$\mathbb{R}P^2$ torus Klein bottle

define CW structures on the quotient surface with a single 2-cell.

e.g. $\mathbb{R}P^2$: $(\mathbb{R}P^2)^{(0)} = *$ $(\mathbb{R}P^2)^{(1)} = S^1$, $(\mathbb{R}P^2)^{(2)} = \mathbb{R}P^2$

with attaching map $S^1 \longrightarrow (\mathbb{R}P^2)^{(1)} = S^1$
$z \longmapsto z^2$

- if X is a simplicial set, then $|X|$ is a CW complex with

$$|X|^{(n)} = \left(\coprod_{k \leq n} X_k \times \Delta^k \right) / \sim \subset |X|$$

The n-cells correspond to the non-degenerate n-simplices of X.
Moreover, if $f: X \longrightarrow Y$ is a simplicial map, then $|f|: |X| \longrightarrow |Y|$ is cellular.

- If B is a CW complex and $p: E \longrightarrow B$ is a covering, then E has a CW structure with $E^{(n)} = p^{-1}(B^{(n)})$. If E is connected and $e \in E$, $b = p(e)$, then $\pi_1(B, b)$ acts on E via cellular maps. (Exercise 10.2)

- $\mathbb{RP}^n$ has a CW structure with one cell in each dimension $\leq n$ (Exercise 10.3)
  $\mathbb{CP}^n$ has a CW structure with one cell in each <u>even</u> dimension $\leq 2n$

<u>Facts about CW complexes</u> (see Hatcher, Appendix A)

- CW complexes are:
  - Hausdorff, even normal
  - paracompact (every open cover admits a locally finite refinement)
  - locally contractible (every point has a basis of open contractible neighborhoods)
    In particular, they satisfy condition (†) from covering space theory
  - compactly generated (see Exercise 10.1)