There is a canonical functor Top $\longrightarrow$ hTop, which is the identity on objects, and the quotient map $\mathrm{Hom}(X,Y) \longrightarrow [X,Y]$ on hom-sets.

<u>Definition</u>.

- A map $f : X \to Y$ in Top is a <u>homotopy equivalence</u> if it becomes an isomorphism in hTop. Equivalently: there exists $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.
- Two spaces X and Y are <u>homotopy equivalent</u> (or have the <u>same homotopy type</u>), written $X \simeq Y$, if there exists a homotopy equivalence between them.
- A space X is <u>contractible</u> if $X \simeq \underset{\text{1-pt space}}{*}$.

<u>Remarks</u>
- Every homeomorphism is a homotopy equivalence.
- $\simeq$ is an equivalence relation on $\mathrm{Ob}(\mathrm{Top})$ ($\Longleftrightarrow$ $\cong$ in $\mathrm{Ob}(\mathrm{hTop})$).

<u>Examples</u>
- $\mathbb{R}^n$ and D^n are contractible for all $n \geq 0$.

$$\{0\} \underset{p}{\overset{i}{\rightleftarrows}} \mathbb{R}^n \qquad p \circ i = \mathrm{id}_{\{0\}}$$
$$i \circ p \simeq \mathrm{id}_{\mathbb{R}^n} \text{ via } H : \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n, \quad (x,t) \longmapsto tx$$

The same H works for D^n.

- More generally, if $X \subset \mathbb{R}^n$ is <u>star-shaped</u> (i.e.: $\exists\, p \in X$ such that $\forall\, x \in X$, the line segment from p to x is contained in X), then X is contractible.
- $S^1 \simeq \mathbb{R}^2 \setminus \{0\}$:

$$S^1 \underset{p}{\overset{i}{\rightleftarrows}} \mathbb{R}^2 \setminus \{0\} \qquad p(x) = \frac{x}{\|x\|}$$

$$p \circ i = \mathrm{id}_{S^1}$$
$$i \circ p \simeq \mathrm{id}_{\mathbb{R}^2 \setminus \{0\}} \text{ via } H : (\mathbb{R}^2 \setminus \{0\}) \times I \longrightarrow \mathbb{R}^2 \setminus \{0\}, \quad (x,t) \longmapsto tx + (1-t)\cdot\frac{x}{\|x\|}$$

- $\infty \simeq$ 
$\simeq$ 
$\simeq$  $\simeq \mathbb{R}^2 \setminus 2$ points
$\simeq \mathbb{R}^3 \setminus 2$ disjoint lines

Alternative definition of homotopy

Definition Let C be a category with finite products, let $X, Y \in C$.
An exponential of Y by X is an object $Y^X \in C$ together with a morphism $ev : Y^X \times X \longrightarrow Y$ such that:

$\forall Z \in C$, the map

$$\mathrm{Hom}(Z \times X, Y) \longleftarrow \mathrm{Hom}(Z, Y^X)$$
$$\left(Z \times X \xrightarrow[f \times id_X]{} Y^X \times X \xrightarrow[ev]{} Y\right) \longmapsfrom (f : Z \longrightarrow Y^X)$$

is bijective.

If it exists, the pair (Y^X, ev) is unique up to unique isomorphism.

Fact: If X is a locally compact top. space and Y is any top. space, then the exponential Y^X exists in Top. It is the set of continuous maps $\mathrm{Hom}_{Top}(X, Y)$ equipped with the compact-open topology, which is the topology generated by the subsets

every point has a basis of compact neighborhoods.

$$\{f : X \to Y \mid f(K) \subset U\} \subset \mathrm{Hom}_{Top}(X, Y)$$

for every $K \subset X$ compact and $U \subset Y$ open.

Since I is locally compact, $\mathrm{Hom}(X \times I, Y) \cong \mathrm{Hom}(X, Y^I)$ for all $X, Y \in Top$.

So we can think of a homotopy as a continuous map $X \longrightarrow Y^I$
$x \longmapsto H(x,-) : I \to Y$.

If X is locally compact, then $\mathrm{Hom}(X \times I, Y) \cong \mathrm{Hom}(I, Y^X)$ for all $Y \in Top$.

That is, a homotopy is equivalently a path in Y^X: $t \mapsto H(-,t) : X \to Y$.

Hence, $[X, Y]$ is the set of path-connected components of Y^X.

Relative homotopy

Definition $f, g : X \to Y$ continuous maps, $A \subset X$ subset s.t. $f|_A = g|_A$.
A homotopy rel. A from f to g is a homotopy $H : X \times I \longrightarrow Y$ from f to g such that $H(a,t) = f(a)$ for all $a \in A$ and $t \in I$. In other words:

$$\begin{tikzcd}
A \times I \arrow[r, "pr_1"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "f|_A = g|_A"] \\
X \times I \arrow[r, "H"'] & Y.
\end{tikzcd}$$

Notation: $f \simeq_A g$

Special case 1: pointed homotopy

- Top_* = category of pointed topological spaces:

$$\begin{cases} \text{objects are pairs } (X, x_0) \text{ with } X \in Top,\ x_0 \in X \quad \text{"base point"} \\ \text{morphisms: } (X,x_0) \xrightarrow[f]{} (Y,y_0) \text{ is a continuous map } f: X \to Y \\ \qquad \text{such that } f(x_0) = y_0. \end{cases}$$

Given $f, g: (X,x_0) \to (Y,y_0)$, a homotopy rel $\{x_0\}$ from f to g is called a pointed homotopy: $H: X \times I \to Y$ s.t.

$$\begin{cases} H(x,0) = f(x) \\ H(x,1) = g(x) \\ H(x_0,t) = y_0 \end{cases}$$

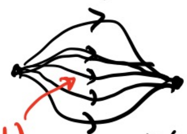
- $[(X,x_0),(Y,y_0)]_* = \mathrm{Hom}_{Top_*}((X,x_0),(Y,y_0)) / \underbrace{\simeq_*}_{\text{pointed homotopy}}$

These are the hom-sets of a category $hTop_*$, called the pointed homotopy category.

Special case 2: path-homotopy

Recall that a path in X is a continuous map $\alpha: I \to X$. Given paths $\alpha, \beta: I \to X$ with the same endpoints $(\alpha(0) = \beta(0), \alpha(1) = \beta(1))$, a path-homotopy from α to β is a homotopy rel. $\{0,1\}$. We write $\alpha \simeq_p \beta$.

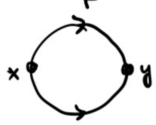
$\beta = H(-,1)$



$H: I \times I \to X$.

$H(-,t)$ $\quad \alpha = H(-,0)$

Example: $X = S^1$



$\alpha, \beta: I \to S^1$ with same endpoints.

α and β are not path-homotopic (proof later)

However, $\alpha \simeq \beta$; $\quad \alpha \simeq$ constant path at x $\simeq \beta$.

Homotopy invariance

Definition: A functor $F: Top \to C$ is homotopy invariant if $F(f) = F(g)$ whenever $f \simeq g$.

Proposition (Universal property of the homotopy category)
Let C be a category. The canonical functor $Top \xrightarrow{h} hTop$ induces a fully faithful functor

$$\begin{aligned} \mathrm{Fun}(hTop, C) &\longrightarrow \mathrm{Fun}(Top, C) \\ F &\longmapsto F \circ h \end{aligned}$$

whose essential image is the full subcategory of homotopy invariant functors.