- a CW complex is
  - compact iff it is finite
  - locally compact iff it is locally finite (every point has a neighborhood intersecting finitely many cells)

- if (X,Y) is a relative CW complex, then Y is a closed neighborhood deformation retract in X. Hence

$$H_*(X,Y) \cong \tilde{H}_*(X/Y).$$

- A <u>subcomplex</u> of a CW complex X is a subspace $Y \subset X$ such that for any cell $C \subset X$, if $C \cap Y \neq \emptyset$, then $\overline{C} \subset Y$.

  For example, $X^{(n)} \subset X$ is a subcomplex.

  Then Y is a CW complex with $Y^{(n)} = X^{(n)} \cap Y$. Moreover (X,Y) is a relative CW complex, with n-skeleton $X^{(n)} \cup Y$.

- (Cellular approximation theorem) If X and Y are CW complexes, every continuous map $X \longrightarrow Y$ is homotopic to a cellular map.

- the interval I has a CW structure with $I^{(0)} = \{0,1\}$, $I^{(1)} = I$.

  For $X \in CW$, $X \times I$ has an induced CW structure s.t. $i_0, i_1 : X \hookrightarrow X \times I$ are cellular (cf. Ex-sem 10.4). Hence, we can define a <u>cellular homotopy</u> between cellular maps. The cellular approximation theorem implies that the forgetful functor $hCW \longrightarrow hTop$ is fully faithful.

<u>CW structures and manifolds</u>

- every smooth manifold admits a CW structure.

- every topological manifold (compact or dim n) is homotopy equivalent to a CW complex (finite or dim n)

- every compact topological manifold of dim $\neq 4$ admits a CW structure.

  In dimension 4, the existence of a CW structure is an open question.

<u>Remark</u> "CW" stands for:

- weak topology: $U \subset X$ is open iff $(\Phi_n^\alpha)^{-1}(U) \subset D^n$ is open for every characteristic map Φ_n^α.
- closure finiteness: the closure of a cell intersects only finitely many cells. This is a special case of the following:

<u>Proposition</u> ("closure finiteness")
Let X be a CW complex and $K \subset X$ a compact subspace. Then K is contained in a finite subcomplex of X. In particular, $K \subset X^{(n)}$ for some n.

<u>Proof</u>. We'll prove:

1) K intersects only finitely many cells.
2) every cell is contained in a finite subcomplex.

1). Let $S \subset X$ be a set of points lying in different cells. We claim that S is closed, i.e., $S \cap X^{(n)} \subset X^{(n)}$ is closed for all n. We prove this by induction on n.

$$(\Phi_n^\alpha)^{-1}(S) = \underbrace{(\varphi_n^\alpha)^{-1}(S)}_{\text{closed in } S^{n-1}} \cup (\text{at most one point}) \text{ is closed in } D^n.$$

Since this applies also to any subset of S, this shows that S is <u>discrete</u>. Let $S \subset K$ be a set of representatives of cells intersecting K. Then S is both discrete and compact, so S is finite.

2) Let $C = \Phi_n^\alpha((D^n)^\circ)$. By induction, $\Phi_n^\alpha(\partial D^n)$ is contained in a finite subcomplex Y. Then $Y \cup C$ is a finite subcomplex. □

<u>Cellular chains</u>

Let X be a CW complex. Then

$$H_k(X^{(n)}, X^{(n-1)}) \cong \tilde{H}_k(X^{(n)}/X^{(n-1)}) \cong \tilde{H}_k\Big(\bigvee_{\alpha \in I_n} S^n\Big) \overset{\text{Ex. 8.3}}{\cong} \begin{cases} \bigoplus_{\alpha \in I_n} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

We define a chain complex $C_*^{cell}(X) \in Ch_{\geq 0}(Ab)$ as follows:

- $C_n^{cell}(X) = H_n(X^{(n)}, X^{(n-1)}) \qquad \left(\cong \bigoplus_{n\text{-cells}} \mathbb{Z}\right)$
- the differential $d : C_n^{cell}(X) \longrightarrow C_{n-1}^{cell}(X)$ is the composite:

$$\begin{tikzcd}
H_n(X^{(n)}, X^{(n-1)}) \arrow[r, "\partial"] & H_{n-1}(X^{(n-1)}) \arrow[r, "id_*"] \arrow[dr, "0"', "\text{(LES of pair)}"] & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \arrow[d, "\partial"] \\
 & & H_{n-2}(X^{(n-2)}) \arrow[d, "id_*"] \\
 & & H_{n-2}(X^{(n-2)}, X^{(n-3)})
\end{tikzcd}$$

$d^2 = 0$:

<u>Definition</u> $C_*^{cell}(X)$ is called the <u>cellular chain complex</u> of X.
Its homology is called the <u>cellular homology</u> of X.

<u>Functoriality</u>: if $f : X \to Y$ is a cellular map, then it induces a morphism of pairs $(X^{(n)}, X^{(n-1)}) \longrightarrow (Y^{(n)}, Y^{(n-1)})$ and hence

$$C_n^{cell}(X) \xrightarrow{f_*} C_n^{cell}(Y).$$

Moreover, f_* is a chain map, since ∂ in the LES of a pair is natural in the pair.

$$\Longrightarrow \quad C_*^{cell} : CW \longrightarrow Ch_{\geq 0}(Ab).$$

<u>Theorem</u> (cellular homology = singular homology)
Let X be a CW complex, A an abelian group, $n \in \mathbb{Z}$. There is an isomorphism

$$H_n(C_*^{cell}(X) \otimes A) \cong H_n(X, A)$$

natural in X and A.

<u>Lemma</u> Let X be a CW complex. Then:

1) $H_*(X) \xleftarrow{\cong} \operatorname{colim}_{n \to \infty} H_*(X^{(n)})$
2) For $k > n$, $H_k(X^{(n)}) = 0$
3) The inclusion $X^{(n)} \hookrightarrow X$ induces an iso $H_k(X^{(n)}) \xrightarrow{\cong} H_k(X)$ for $k < n$.

Proof. 1) $\mathrm{Hom}_{Top}(\Delta^k, X) \xleftarrow{\cong} \operatorname{colim}_{n\to\infty} \mathrm{Hom}_{Top}(\Delta^k, X^{(n)})$ since Δ^n is compact

$\Rightarrow H_*(X) \cong \operatorname{colim}_{n\to\infty} H_*(X^{(n)})$ as before.

3) By 1) it suffices to show that $X^{(n)} \hookrightarrow X^{(n+m)}$ is an iso on $H_{<n}$. By induction on m, we can assume $m=1$. LES of pair:

$$\cdots \to \overset{=0 \text{ if } k\neq n}{H_{k+1}(X^{(n+1)}, X^{(n)})} \xrightarrow{\partial} H_k(X^{(n)}) \to H_k(X^{(n+1)}) \to \overset{=0 \text{ if } k\neq n+1}{H_k(X^{(n+1)}, X^{(n)})} \xrightarrow{\partial} H_{k-1}(X^{(n)}) \to \cdots$$

$\Rightarrow H_k(X^{(n)}) \to H_k(X^{(n+1)})$ is iso if $k \neq n, n+1$. $(*)$

2) $H_k(X^{(n)}) \leftarrow H_k(X^{(n-1)}) \leftarrow \cdots \leftarrow H_k(X^{(-1)}) = 0$

By $(*)$, all these maps are iso if $k > n$. □

Proof of theorem We look at the LES of the pairs $(X^{(n+1)}, X^{(n)})$
$(X^{(n)}, X^{(n-1)})$
$(X^{(n-1)}, X^{(n-2)})$

$$\begin{tikzcd}
 & & & H_n(X^{(n+1)},X^{(n)})=0 & \\
 & & H_n(X^{(n+1)}) \cong H_n(X) \arrow[ur] & & \\
0=H_n(X^{(n-1)}) \arrow[dr] & & & & \\
 & H_n(X^{(n)}) \arrow[uur] \arrow[dr, hook] & & & \\
H_{n+1}(X^{(n+1)},X^{(n)}) \arrow[ur, "\partial"] \arrow[rr, "d"] & & H_n(X^{(n)},X^{(n-1)}) \arrow[rr, "d"] \arrow[dr, "\partial"] & & H_{n-1}(X^{(n-1)},X^{(n-2)}) \\
 & & & H_{n-1}(X^{(n-2)}) \arrow[ur, hook] & \\
 & & 0=H_{n-1}(X^{(n-2)}) \arrow[ur] & &
\end{tikzcd}$$

(by the lemma.)

$\Rightarrow H_n(X) \cong H_n(X^{(n)}) / \mathrm{im}(\partial) \cong \ker(\partial)/\mathrm{im}(d)$
$= \ker(d)/\mathrm{im}(d) = H_n(C_*^{cell}(X))$.

With coeff. in A, note that $H_*(X^{(n)}, X^{(n-1)}, A) \cong H_*(X^{(n)}, X^{(n-1)}) \otimes A$.
$\left(\cong \bigoplus_{n\text{-cells}} A.\right)$ □