Corollary Let X be a CW complex with finitely many n-cells. Then $H_n(X)$ is a finitely generated abelian group.

Proof. $H_n(X)$ is a quotient of $\ker\Big(d : \bigoplus_{n\text{-cells}} \mathbb{Z} \longrightarrow \bigoplus_{(n-1)\text{-cells}} \mathbb{Z}\Big)$

Any subgroup of $\mathbb{Z}^a$ is generated by $\leq a$ elements. $\square$

If (X,Y) is a relative CW complex, we can define $C_*^{cell}(X,Y)$ with

$$C_n^{cell}(X,Y) = H_n(X^{(n)}, X^{(n-1)}) \quad \Big(\cong \bigoplus_{n\text{-cells}} \mathbb{Z}\Big)$$

with d as before.

Corollary For (X,Y) a relative CW complex and A an abelian group

$$H_n(C_*^{cell}(X,Y) \otimes A) \cong H_n(X,Y,A).$$

Proof X/Y is a CW complex with $(X/Y)^{(n)} = X^{(n)}/Y$

$X^{(0)} = Y \amalg 0\text{-cells} \implies (X/Y)^{(0)} = Y/Y \amalg (X^{(0)} \setminus Y)$

$$\begin{tikzcd}
\cdots \arrow[r] & C_1^{cell}(X,Y) \arrow[r] & C_0^{cell}(X,Y) \arrow[r] \arrow[r, phantom, shift right=4ex, "\overset{0}{\underset{Y/Y}{\uparrow}}"] & 0 \\
\cdots \arrow[r] \arrow[u, equal] & C_1^{cell}(X/Y) \arrow[r] \arrow[u, equal] & C_0^{cell}(X/Y) \arrow[r, "\varepsilon"'] \arrow[u] & \mathbb{Z} \arrow[u] \\
& 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r, "\cong"] \arrow[u] & \mathbb{Z} \arrow[u, equal]
\end{tikzcd}$$

short exact sequence of chain complexes

$$H_n(C_*^{cell}(X,Y)) \overset{\text{LES}}{\cong} \begin{cases} H_n(C_*^{cell}(X/Y)) & \text{if } n \geq 1 \\ \ker(\varepsilon : H_0 \to \mathbb{Z}) & \text{if } n = 0 \end{cases} \overset{\text{cell}=\text{sing}}{\cong} \tilde{H}_n(X/Y) \overset{\text{closed NDR}}{\cong} H_n(X,Y).$$

$\square$

Example $\mathbb{C}P^n$ has a CW structure with one cell in each even dimension $\leq 2n$

$\rightsquigarrow C_*^{cell}(\mathbb{C}P^n)$: $\quad \underset{2n}{\mathbb{Z}} \to 0 \to \underset{2n-2}{\mathbb{Z}} \to 0 \to \mathbb{Z} \cdots \to \underset{1}{0} \to \underset{0}{\mathbb{Z}}$

$$\implies H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq 2n \text{ even.} \\ 0 & \text{otherwise.} \end{cases}$$

Computing cellular chains

Let X be a CW complex with characteristic maps $\Phi^n_\alpha : D^n \longrightarrow X^{(n)}$,
$\alpha \in I_n$

$$\begin{tikzcd}
C^{cell}_n(X) \arrow[r, "d"] & C^{cell}_{n-1}(X) \\
\bigoplus\limits_{\alpha\in I_n} \mathbb{Z} \arrow[u, phantom, "\cong" sloped] \arrow[r, "?"] & \bigoplus\limits_{\beta\in I_{n-1}} \mathbb{Z} \arrow[u, phantom, "\cong" sloped]
\end{tikzcd}$$

depends on $H_n(D^n/\partial D^n)\cong\mathbb{Z}$

<u>Remark</u>: In a chain complex, if you change the signs of some differentials, the homology does not change.

$\Rightarrow$ we don't care about the sign of d.

For $\alpha\in I_n$ and $\beta\in I_{n-1}$, let $d_{\alpha\beta}$ be the degree of the following map $S^{n-1}\longrightarrow S^{n-1}$:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\varphi^n_\alpha"] & X^{(n-1)} \arrow[r, "\text{collapse}"] \arrow[dr, "q_\beta"'] & X^{(n-1)}/\left(X^{(n-1)} \setminus \Phi^{n-1}_\beta\left((D^{n-1})^\circ\right)\right) \\
& & D^{n-1}/\partial D^{n-1} \cong S^{n-1} \arrow[u, "\Phi^{n-1}_\beta", "\cong"']
\end{tikzcd}$$

<u>Proposition</u> (cellular boundary formula)

Up to a sign, the differential $d: \bigoplus\limits_{\alpha\in I_n} \mathbb{Z}\alpha \longrightarrow \bigoplus\limits_{\beta\in I_{n-1}} \mathbb{Z}\beta$

is given by $d(\alpha) = \sum\limits_{\beta\in I_{n-1}} d_{\alpha\beta}\cdot\beta$.

<u>Proof</u>: Recall: if $(X_i)_{i\in J}$ are "nicely" pointed spaces, then

induced by $X_j \hookrightarrow \bigvee\limits_{i\in J} X_i$

$$\tilde H_*\Big(\bigvee_{i\in J} X_i\Big) \cong \bigoplus_{i\in J} \tilde H_*(X_i)$$

induced by $\bigvee\limits_{i\in J} X_i \twoheadrightarrow X_j$ sends X_i, $i\neq j$, to the base point.

$$\begin{tikzcd}
 & & & \tilde H_{n-1}(S^{n-1}) & \\
\mathbb{Z}\alpha \arrow[r, phantom, "\cong"] \arrow[d, hook] & \tilde H_n(D^n/\partial D^n) \arrow[r, "\partial", "\cong"'] \arrow[d, "(\Phi^n_\alpha)_*"] \arrow[dr, phantom, "\circlearrowright"] & \tilde H_{n-1}(\partial D^n) \arrow[ur, "\cdot d_{\alpha\beta}"] \arrow[d, "(\varphi^n_\alpha)_*"] \arrow[dr, phantom, "\circlearrowright \text{ by def of } d_{\alpha\beta}"] & \tilde H_{n-1}(D^{n-1}/\partial D^{n-1}) \arrow[u, phantom, "\cong" sloped] \arrow[r, phantom, "\cong"] & \mathbb{Z}\beta \\
\bigoplus\limits_{\alpha\in I_n}\mathbb{Z}\alpha \arrow[r, phantom, "\cong"] & \tilde H_n(X^{(n)}/X^{(n-1)}) \arrow[r, "\partial"] \arrow[rr, bend right, "d"'] & \tilde H_{n-1}(X^{(n-1)}) \arrow[r] & \tilde H_{n-1}(X^{(n-1)}/X^{(n-2)}) \arrow[u, "(q_\beta)_*"] \arrow[r, phantom, "\cong"] & \bigoplus\limits_{\beta\in I_{n-1}}\mathbb{Z}\beta \arrow[u, two heads]
\end{tikzcd}$$

$\square$

Def (Local degree) Let $n \geq 1$, $f: S^n \longrightarrow S^n$ a continuous map, $x \in S^n$, $y = f(x)$.
Suppose there exists a nhood U of x such that $f(U - \{x\}) \subset S^n - \{y\}$.
The <u>local degree</u> of f at x is $dg_x(f) \in \mathbb{Z}$ such that:

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\cong"] \arrow[dr, "\cdot dg_x(f)"'] & H_n(S^n, S^n - \{x\}) & H_n(U, U - \{x\}) \arrow[l, "\cong"', "\text{excision}"] \arrow[d, "f_*"] \\
 & H_n(S^n) \arrow[r, "\cong"] & H_n(S^n, S^n - \{y\})
\end{tikzcd}$$

Prop (Local degree formula).
$f: S^n \longrightarrow S^n$, $y \in S^n$ such that $f^{-1}(y) = \{x_1, \dots, x_k\}$.
Then $dg(f) = \sum_{i=1}^{k} dg_{x_i}(f)$.

Rmk: If $f: S^n \longrightarrow S^n$ is smooth, Sard's theorem implies that there exists y with $f^{-1}(y) = \{x_1, \dots, x_k\}$ and $dg_{x_i}(f) = \pm 1$.

Proof: Choose nhoods U_i of x_i s.t. $U_i \cap U_j = \emptyset$.

$$\begin{tikzcd}
 & \bigoplus_{i=1}^{k} H_n(S^n) \arrow[drr, bend left, "\cong"] & & \\
H_n(S^n) \arrow[ur, "\Delta"] \arrow[r] \arrow[d, "f_*"'] & H_n(S^n, S^n - f^{-1}\{y\}) \arrow[d, "f_*"] & H_n\left(\coprod_{i=1}^{k} U_i, \coprod_{i=1}^{k} (U_i - \{x_i\})\right) \arrow[l, "\cong"', "\text{exc.}"] \arrow[r, phantom, "\cong"] & \bigoplus_{i=1}^{k} H_n(U_i, U_i - \{x_i\}) \arrow[dll, bend left, "\Sigma (f|_{U_i})_*"] \\
H_n(S^n) \arrow[r, "\cong"] & H_n(S^n, S^n - \{y\}) & &
\end{tikzcd}$$

$\square$

Examples

- Σ_g

4g-gon

$\rightsquigarrow$ CW structure on Σ_g with
1 0-cell
$2g$ 1-cells
1 2-cell.

with attaching map $S^1 \longrightarrow \Sigma_g^{(1)} = \bigvee_{2g} S^1$

given by $[a_1, b_1] \cdot [a_2, b_2] \cdots [a_g, b_g]$ in $\pi_1(\bigvee_{2g} S^1) = \langle a_1, b_1, \dots, a_g, b_g \rangle$

$S^1 \xrightarrow{\text{attach}} \bigvee_{2g} S^1 \xrightarrow{q_{a_1}} S^1$

$(q_{a_1})_*([a_1, b_1] \cdots [a_g, b_g]) = 0$

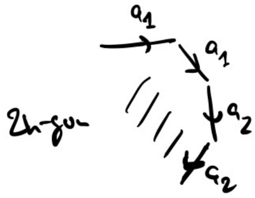
on π_1: $(q_{a_1})_*: \langle a_1, b_1, \dots, a_g, b_g \rangle \longrightarrow \mathbb{Z}$, $a_1 \mapsto 1$, other $\mapsto 0$

$\Rightarrow$ the map is null-homotopic

$$\Rightarrow \quad C_*^{cell}(\Sigma_g): \qquad \mathbb{Z} \xrightarrow{0} \bigoplus_{2g} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$\Rightarrow \quad H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } k=0,2 \\ \mathbb{Z}^{2g} & \text{if } k=1 \\ 0 & \text{otherwise.} \end{cases}$$

- N_h



CW structure with 1 0-cell, h 1-cell, 1 2-cell

$$\Rightarrow \quad C_*^{cell}(N_h): \qquad \mathbb{Z} \xrightarrow{d} \mathbb{Z}^h \xrightarrow{0} \mathbb{Z}$$

attaching map $S^1 \to N_h^{(1)} = \bigvee_h S^1$ is $a_1^2 a_2^2 \cdots a_h^2$ in $\pi_1(\bigvee_h S^1) = \langle a_1, \dots, a_h \rangle$.

$$S^1 \xrightarrow{\text{attach}} \bigvee_h S^1 \xrightarrow{q_{a_i}} S^1 \qquad (q_{a_i})_*(a_1^2 a_2^2 \cdots a_h^2) = a_i^2 \in \pi_1(S^1) = \langle a_i \rangle$$
$$= 2 \in \mathbb{Z}$$

$\Rightarrow$ degree is 2

so $d: \mathbb{Z} \hookrightarrow \bigoplus_h \mathbb{Z}$

$1 \longmapsto (2, 2, \dots, 2)$

$\bigoplus_h \mathbb{Z}$ has basis $(1, 1, \dots, 1), e_1, \dots, e_{h-1}$

$\Rightarrow \operatorname{coker}(d) = \mathbb{Z}/2 \oplus \mathbb{Z}^{h-1}$.

$$\Rightarrow \quad H_k(N_h) = \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}^{h-1} & \text{if } k=1 \\ 0 & \text{otherwise.} \end{cases}$$