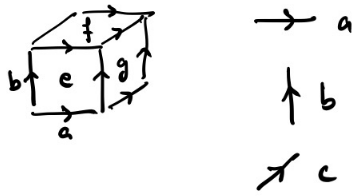
• $T^3 = S^1 \times S^1 \times S^1$



CW structure on T^3 with
1 0-cell
3 1-cells
3 2-cells
1 3-cell

$C_*^{cell}(T^3)$: $\mathbb{Z} \xrightarrow[(**)]{0} \mathbb{Z}e \oplus \mathbb{Z}f \oplus \mathbb{Z}g \xrightarrow[(*)]{0} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{0} \mathbb{Z}$

(*)
$$\begin{tikzcd} S^1 \arrow[r] & (T^3)^{(1)} \cong S^1 \vee S^1 \vee S^1 \arrow[r, "a"] \arrow[r, "b"'] \arrow[dr, "c"'] & S^1 \\ & & S^1 \end{tikzcd}$$
$S^1 = \partial e$

$[a,b] \in \pi_1(S^1 \vee S^1 \vee S^1) = \langle a, b, c \rangle$ $\Rightarrow$ degree is 0

(**)
$$S^2 \twoheadrightarrow (T^3)^{(2)} \xrightarrow{\text{collapse complement of } e} S^2 = I^2/\partial I^2$$
$S^2 = \partial I^3$

• pick a point in the interior of the front face e
this point has two preimages, with local degrees of opposite signs

$$\begin{tikzcd} & \text{front face}/\partial \arrow[dr, phantom, "\cong"] & \\ \partial I^3 \arrow[ur] \arrow[dr] & \text{differ by a reflection} & I^2/\partial I^2 \\ & \text{back face}/\partial \arrow[ur, phantom, "\cong"] & \end{tikzcd}$$

By the local degree formula, $\deg = 1 - 1 = 0$.

The Euler characteristic

Def. Let A be an abelian group. The rank of A is the dimension of the $\mathbb{Q}$-vector space $A \otimes \mathbb{Q}$: $\operatorname{rk}(A) = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$

Def. • Let $C_* \in Ch(Ab)$ be a chain complex. The rank or Euler characteristic of C_* is
$$\chi(C_*) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rk} H_i(C_*)$$
defined when $\operatorname{rk} H_i(C)$ is zero for all but finitely many i and finite for all i.

- Let X be a topological space. The <u>Euler characteristic</u> of X is
$$\chi(X) = \chi(C_*(X))$$

<u>Proposition</u>

1) Let C_* be a bounded chain complex of finitely generated abelian groups. ($C_i = 0$ for all but finitely many i)

Then $$\chi(C_*) = \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{rk} C_i .$$

2) If $0 \to C_* \to D_* \to E_* \to 0$ is a SES of chain complexes, then $\chi(D_*) = \chi(C_*) + \chi(E_*)$.

<u>Proof</u> Exercise.

<u>Corollary</u> Let X be a finite CW complex. Then $\chi(X)$ is defined and
$$\chi(X) = \sum_{i \geq 0} (-1)^i \# \{i\text{-cells}\}$$

<u>Pf.</u> $$\chi(C_*(X)) = \chi(C_*^{cell}(X)) = \sum_{i \geq 0} (-1)^i \underbrace{\operatorname{rk} C_i^{cell}(X)}_{\#\, i\text{-cells}} . \qquad \square$$

<u>Remark</u>: The rank of $H_i(X)$ is called the i-th Betti number of X, $b_i(X)$.
Thus, $$\chi(X) = \sum_{i \geq 0} (-1)^i b_i(X) .$$

<u>Examples</u>:

- $\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$
- $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$ e.g. $\chi(T) = 0$
- $\chi(N_h) = 1 - h + 1 = 2 - h$ e.g. $\chi(\mathbb{R}P^2) = 1$, $\chi(\text{Klein bottle}) = 0$
- For any convex polyhedron, then
$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = \chi(S^2) = 2$$
(Euler's formula)
- $\chi(\mathbb{R}P^n) = \underbrace{1 - 1 + 1 - 1 \cdots \pm 1}_{n+1} = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$
- $\chi(\mathbb{C}P^n) = 1 - 0 + 1 - 0 + 1 \cdots = n+1$

Properties

1) $\chi(X \sqcup Y) = \chi(X) + \chi(Y)$

2) $\chi(X \times Y) = \chi(X)\,\chi(Y)$

3) If $E \xrightarrow{p} B$ is a covering of degree d (i.e. $\# p^{-1}(b) = d$ for all $b \in B$)
then $\chi(E) = d\,\chi(B)$

4) a) if $X = U^\circ \cup V^\circ$, then $\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$

b) if
$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, hook] & X' \end{tikzcd}$$
is a pushout square where $Y \subset X$, $Y' \subset X'$ are closed NDRs (e.g. (X,Y) is a relative CW complex)
then $\chi(X') = \chi(X) + \chi(Y') - \chi(Y)$.

Proof.

1) follows from additivity axiom : $H_*(X \sqcup Y) \cong H_*(X) \oplus H_*(Y)$.

2) (X, Y finite CW complexes). Let $a_n = \#$ n-cells of X
$b_n =$ ——— Y.

$X \times Y$ has a CW structure with n-cells $= \coprod_{p+q=n} \{\text{p-cells of } X\} \times \{\text{q-cells of } Y\}$
(Exercises 10)

$$\Rightarrow \chi(X \times Y) = \sum_n (-1)^n \#(\text{n-cells of } X \times Y) = \sum_n (-1)^n \sum_{p+q=n} a_p \cdot b_q = \Big(\sum_p (-1)^p a_p\Big)\Big(\sum_q (-1)^q b_q\Big)$$
$$= \chi(X)\,\chi(Y).$$

3) (B finite CW complex)

E has a CW structure with # n-cells $= d \cdot$ (# n-cells in B)

4) a) We have a short exact sequence

$$0 \to C_*(U \cap V) \to C_*(U) \oplus C_*(V) \to C_*^{\{U,V\}}(X) \to 0$$
$\cap$ quasi-iso (excision)
$C_*(X)$

$$\Rightarrow \chi(C_*(U) \oplus C_*(V)) = \chi(C_*(U \cap V)) + \chi(C_*(X))$$
$$\chi(U) + \chi(V) \quad = \quad \chi(U \cap V) + \chi(X).$$

b) $$0 \to C_*(Y) \to C_*(X) \to C_*(X,Y) \to 0$$
$\downarrow$ quasi-iso (closed NDR)

$$C_*(X/Y, Y/Y) \cong C_*(X'/Y', Y'/Y')$$

$$\chi(X) - \chi(Y) = \chi(\ -\ ) = \chi(X') - \chi(Y').$$ $\square$

§8. Künneth, universal coefficients, cohomology

<u>Definition</u> Let R be a comm. ring, $(C_*, d_C), (D_*, d_D) \in Ch(Mod_R)$. We define:

- $C_*[i]$ is the chain complex with $C_*[i]_n = C_{n-i}$ and differential $(-1)^i d_C$
- $C_* \otimes_R D_*$ is the chain complex with

$$(C_* \otimes_R D_*)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q$$

$$d(a \otimes b) = d_C(a) \otimes b + (-1)^{\deg(a)} a \otimes d_D(b)$$

- $\underline{Hom}_R(C_*, D_*)$ is the chain complex with

$$\underline{Hom}_R(C_*, D_*)_n = \prod_{i \in \mathbb{Z}} Hom_R(C_i, D_{i+n})$$

$$d((f_i)_{i\in\mathbb{Z}}) = (d_D \circ f_i - (-1)^n f_{i-1} \circ d_C)_{i \in \mathbb{Z}}$$

$$f_i : C_i \to D_{i+n}$$

<u>Remark</u>: • $Z_n(\underline{Hom}_R(C_*, D_*)) = Hom_{Ch(Mod_R)}(C_*[n], D_*)$

$$\Downarrow$$

$$H_n(\underline{Hom}_R(C_*, D_*)) = Hom_{Ch(Mod_R)}(C_*[n], D_*) / \text{chain homotopy.}$$

• $\otimes_R$ is a symmetric monoidal structure on $Ch(Mod_R)$ with internal hom $\underline{Hom}_R$:

$$\underline{Hom}_R(C_*, \underline{Hom}_R(D_*, E_*)) \cong \underline{Hom}_R(C_* \otimes_R D_*, E_*)$$

<u>Def.</u> Let X be a top. space / simplicial set, A an abelian group.

- The <u>cochains</u> of X with coefficients in A are

$$C^*(X, A) = \underline{Hom}(C_*(X), \underset{\substack{\uparrow \\ \text{in degree } 0.}}{A}) \quad \text{with} \quad \begin{array}{l} C^n(X,A) := C^*(X,A)_{-n} \\ d : C^n(X,A) \to C^{n+1}(X,A) \end{array}$$

- The <u>cohomology</u> of X with coefficients in A is

$$H^n(X, A) = H^n(C^*(X,A)) = H_{-n}(C^*(X,A)) = H_{-n}(\underline{Hom}(C_*(X), A)).$$