__Theorem__ (Eilenberg–Zilber)

Let A and B be simplicial abelian groups. Then there is a quasi-isomorphism

$$\nabla_{A,B} : C_*(A) \otimes C_*(B) \longrightarrow C_*(\underbrace{A \otimes B}_{[n] \mapsto A_n \otimes B_n})$$

natural in A and B.

__Corollary__: Let $X, Y \in \mathrm{Top}$. There is a quasi-isomorphism

$$C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y).$$

__Pf.__ $\mathrm{Sing}(X \times Y) \cong \mathrm{Sing}(X) \times \mathrm{Sing}(Y)$

$$\mathbb{Z}[\mathrm{Sing}(X) \times \mathrm{Sing}(Y)] \cong \mathbb{Z}[\mathrm{Sing}(X)] \otimes \mathbb{Z}[\mathrm{Sing}(Y)].$$

$$\Rightarrow C_*(X \times Y) \overset{\text{def}}{=} C_*(\mathbb{Z}[\mathrm{Sing}(X \times Y)]) \cong C_*(\mathbb{Z}[\mathrm{Sing}\, X] \otimes \mathbb{Z}[\mathrm{Sing}\, Y])$$

$$\uparrow \text{ quasi-iso E-Z}$$

$$C_*(X) \otimes C_*(Y). \qquad \square$$

To compute $H^*(X)$ or $H_*(X \times Y)$, we need to understand H_* of a $\otimes$ or $\underline{\mathrm{Hom}}$ of chain complexes. In general, H_* does not commute with $\otimes$ and $\underline{\mathrm{Hom}}$, because $\otimes$ and $\underline{\mathrm{Hom}}$ are not __exact__ functors.

__Recollections from homological algebra.__

- An R-module M is __flat__ if $M \otimes_R (-) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is exact.

  free $\Rightarrow$ projective $\Rightarrow$ flat

  projective $\Updownarrow$ direct summand of free

- There are functors

$$\mathrm{Tor}_i^R : \mathrm{Mod}_R \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R \qquad i \geq 0$$

  with the following properties:
  - $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$
  - $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(N,M)$.

- If $P_* \to M$ is a projective resolution (more generally, a flat resolution) (i.e. $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ exact sequence with P_i projective / flat), then

$$\mathrm{Tor}_i^R(M,N) \cong H_i(P_* \otimes_R N)$$

- In particular, if M is flat then $\mathrm{Tor}_i^R(M,N) = 0$ $\forall i>0, \forall N$.

- If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence then there is a LES

$$\cdots \to \mathrm{Tor}_2^R(M_3,N) \to \mathrm{Tor}_1^R(M_1,N) \to \mathrm{Tor}_1^R(M_2,N) \to \mathrm{Tor}_1^R(M_3,N) \to M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0$$

- Dually, we have functors

$$\mathrm{Ext}_R^i : \mathrm{Mod}_R^{op} \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R \qquad i \geq 0$$

such that:

- $\mathrm{Ext}_R^0(M,N) = \mathrm{Hom}_R(M,N)$
- If $P_* \to M$ is a projective resolution, $\mathrm{Ext}_R^i(M,N) = H_{-i}(\underline{\mathrm{Hom}}_R(P_*,N))$
  If $N \to Q_*$ is an injective resolution, $\mathrm{Ext}_R^i(M,N) = H_{-i}(\underline{\mathrm{Hom}}_R(M,Q_*))$
  $(0 \to N \to Q_0 \to Q_{-1} \to Q_{-2} \to \cdots)$
- In particular, $\mathrm{Ext}_R^i(M,N) = 0$ for all $i>0$ if either M is projective or N is injective.

- SES in either variable induces a LES of Ext groups, e.g.

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

$$\rightsquigarrow 0 \to \mathrm{Hom}_R(M_3,N) \to \mathrm{Hom}_R(M_2,N) \to \mathrm{Hom}_R(M_1,N) \to \mathrm{Ext}_R^1(M_3,N) \to \cdots$$

- If R is a PID, every submodule of a free module is free.
  In particular, every R-module M has a projective = free resolution of the form

$$0 \to P_1 \to P_0 \to M \to 0$$

Hence: $\mathrm{Tor}_i^R = 0$ for $i \geq 2$

$\mathrm{Ext}_R^i = 0$ for $i \geq 2$.

(R PID). An R-module is flat iff it is torsion-free ($rm = 0$, $r \neq 0 \Rightarrow m = 0$)
$\rightarrow$ a submodule of a flat module is flat.

(Reference: Hilton–Stammbach, "A course in homological algebra".)

Proposition (Universal coefficients)

Let R be a PID, $N \in \mathrm{Mod}_R$, $C_* \in \mathrm{Ch}(\mathrm{Mod}_R)$.

1) If C_* is degreewise flat, there is a split short exact sequence

$$0 \longrightarrow H_n(C_*) \otimes_R N \longrightarrow H_n(C_* \otimes_R N) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(C_*), N) \longrightarrow 0$$

2) If C_* is degreewise free, there is a split short exact sequence

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(C_*), N) \longrightarrow \underbrace{H^n}_{H_{-n}}(\underline{\mathrm{Hom}}_R(C_*, N)) \longrightarrow \mathrm{Hom}_R(H_n(C_*), N) \to 0$$

In both cases, the sequences are natural, but not the splittings.

Proposition (Künneth formula)

Let R be a PID, $C_*, D_* \in \mathrm{Ch}(\mathrm{Mod}_R)$, C_* degreewise flat. Then there is a split short exact sequence

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*) \longrightarrow H_n(C_* \otimes_R D_*) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*)) \longrightarrow 0$$

Rmk
- Univ coeff for $\otimes$ is a special case of Künneth (with D_* in degree 0)
- There is also a dual Künneth for $H_*(\underline{\mathrm{Hom}}_R(C_*, D_*))$.
- If R is not a PID, we have instead the Künneth spectral sequence

$$\mathrm{Tor}_*^R(H_*(C_*), H_*(D_*)) \Rightarrow H_*(C_* \otimes_R D_*)$$

Proof of Künneth

Special case: the differentials in C_* are zero, so $C_* \cong \bigoplus_{i \in \mathbb{Z}} C_i[i]$

Since $C_i \cong H_i(C_*)$ is flat, $\mathrm{Tor}_1^R(H_p(C_*), H_q(D_*)) = 0$.

The statement follows from the fact that H_* commutes with direct sums and with tensoring with a flat module:

$$\bigoplus_{p+q=n} C_p \otimes_R H_q(D_*) \overset{C_p \text{ flat}}{\cong} \bigoplus_{p+q=n} H_q(C_p \otimes_R D_*)$$
$$\cong \bigoplus_{p+q=n} H_{p+q}(C_p[p] \otimes_R D_*)$$
$$\cong H_n\Big(\underbrace{\bigoplus_{p+q=n} C_p[p]}_{\bigoplus_{p\in\mathbb{Z}} C_p[p]} \otimes_R D_*\Big) \cong H_n(C_* \otimes_R D_*)$$

<u>General case</u> Let $B_i \subset Z_i \subset C_i$ be the boundaries and cycles in C_*.

Since R is a PID, B_i and Z_i are flat R-modules.

$B_* \subset Z_* \subset C_*$ subcomplexes of C_* with zero differential.

We have a short exact sequence of chain complexes

$$0 \to Z_* \hookrightarrow C_* \xrightarrow{d} B_*[1] \to 0$$

$0 \to Z_* \otimes_R D_* \to C_* \otimes_R D_* \to B_*[1] \otimes_R D_* \to 0$ is still exact because B_i is flat

$$\left(\begin{array}{l} 0 \to A \to B \to C \to 0 \text{ with } C \text{ flat, then} \\[1em] 0 = \mathrm{Tor}_1^R(C,D) \to A \otimes_R D \to B \otimes_R D \to C \otimes_R D \to 0 \end{array}\right)$$

$\Rightarrow$ get LES:

$$\begin{tikzcd}[column sep=small]
 & & & \mathrm{coker} \cong \ker \cong \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*)) \arrow[dr, hook] & & \\
\cdots \arrow[r] & H_n(B_* \otimes_R D_*) \arrow[r] & H_n(Z_* \otimes_R D_*) \arrow[r] & H_n(C_* \otimes_R D_*) \arrow[u] \arrow[r] & H_{n-1}(B_* \otimes_R D_*) \arrow[r] & H_{n-1}(Z_* \otimes_R D_*) \arrow[r] & \cdots \\
\bigoplus_{p+q=n} \mathrm{Tor}_1^R(H_p(C_*), H_q(D)) \arrow[r] \arrow[uurrr, bend left] & \bigoplus_{p+q=n} B_p \otimes_R H_q(D_*) \arrow[r] \arrow[u, "\cong"', "\text{by the special case}"] & \bigoplus_{p+q=n} Z_p \otimes_R H_q(D_*) \arrow[r, two heads] \arrow[u, "\cong"] & \bigoplus_{p+q=n} H_p(C_*) \otimes_R H_q(D_*) \arrow[r] \arrow[u, dashed, hook, "\text{same image}"'] & 0 \\
0 \ (Z_p \text{ flat}) \arrow[u] & & & &
\end{tikzcd}$$

Splitness : see Hilton–Stammbach, (V, Thm 2.1) $\square$