Example

$$H_*(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & \text{if } *=0 \\ \mathbb{Z}/2 & \text{if } *=1 \\ 0 & \text{if } *\geq 2 \end{cases}$$

- $H_n(\mathbb{R}P^2, \mathbb{Z}/2) \cong H_n(\mathbb{R}P^2)\otimes \mathbb{Z}/2 \oplus \mathrm{Tor}_1(H_{n-1}(\mathbb{R}P^2), \mathbb{Z}/2)$

$$= \begin{cases} \mathbb{Z}/2 & \text{if } n=0 \\ \mathbb{Z}/2 & \text{if } n=1 \\ \mathbb{Z}/2 & \text{if } n=2 \\ 0 & \text{if } n\geq 3 \end{cases}$$

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$
$$\Big\downarrow \otimes \mathbb{Z}/2$$
$$0 \to \underset{\mathbb{Z}/2}{\overset{\|}{\mathrm{Tor}_1}} \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2 \to 0$$

- $H^n(\mathbb{R}P^2, \mathbb{Z}) \cong \mathrm{Hom}(H_n(\mathbb{R}P^2), \mathbb{Z}) \oplus \mathrm{Ext}^1(H_{n-1}(\mathbb{R}P^2), \mathbb{Z})$

$$= \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n=1 \\ \mathrm{Ext}^1(\mathbb{Z}/2,\mathbb{Z}) = \mathbb{Z}/2 & \text{if } n=2 \\ 0 & \text{if } n\geq 3 \end{cases}$$

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$
$$\Big\downarrow \mathrm{Hom}(-,\mathbb{Z})$$
$$0 \leftarrow \underset{\mathbb{Z}/2}{\overset{\|}{\mathrm{Ext}^1}} \leftarrow \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0 \leftarrow 0$$

- $H_n(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \left(\bigoplus_{p+q=n} H_p(\mathbb{R}P^2)\otimes H_q(\mathbb{R}P^2)\right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(\mathbb{R}P^2), H_q(\mathbb{R}P^2))\right)$

$$\cong \begin{cases} \mathbb{Z} & \text{if } n=0 \\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & \text{if } n=1 \\ \mathbb{Z}/2 & \text{if } n=2 \\ \mathbb{Z}/2 & \text{if } n=3 \\ 0 & \text{if } n\geq 4. \end{cases}$$

- $H_n(\mathbb{R}P^2\times\mathbb{R}P^2, \mathbb{Z}/2)$: can use either
  - universal coefficients
  - Künneth with $R = \mathbb{Z}/2$

$$H_n(\mathbb{R}P^2\times\mathbb{R}P^2, \mathbb{Z}/2) \cong \bigoplus_{p+q=n} H_p(\mathbb{R}P^2, \mathbb{Z}/2) \underset{\mathbb{Z}/2}{\otimes} H_q(\mathbb{R}P^2, \mathbb{Z}/2)$$

$$\cong \begin{cases} \mathbb{Z}/2 & \text{if } n=0 \\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & n=1 \\ \mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 & n=2 \\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & n=3 \\ \mathbb{Z}/2 & n=4 \\ 0 & n\geq 5 \end{cases}$$

Poincaré duality.

Definition Let X be a CW complex, A an abelian group.
The <u>cellular cochain complex</u> of X with coefficients in A is

$$C^*_{cell}(X,A) := \underline{\mathrm{Hom}}(C^{cell}_*(X), A).$$

Thm $H^*(C^*_{cell}(X,A)) \cong H^*(X,A)$

Proof:
- can redo the arguments for homology
- alternatively: since $C_*(X)$ and $C^{cell}_*(X)$ are chain complexes of free abelian groups with isomorphic homology, there exists a chain homotopy equivalence $C_*(X) \rightleftarrows C^{cell}_*(X)$ (Exercise 12.4)
  $\underline{\mathrm{Hom}}(-,A)$ preserves chain homotopies, so we have induced chain homotopy equivalences $C^*(X,A) \leftrightarrows C^*_{cell}(X,A)$. $\square$

Example: $H^*(\mathbb{R}P^2)$:

$$\begin{array}{lccccc} & 2 & & 1 & & 0 \\ C^{cell}_*(\mathbb{R}P^2): & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \end{array}$$

$\underline{\mathrm{Hom}}(-,\mathbb{Z})$

$$\begin{array}{lccccc} C^*_{cell}(\mathbb{R}P^2): & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} \\ & 0 & & -1 & & -2 \\ H^*: & \mathbb{Z} & & 0 & & \mathbb{Z}/2 \end{array}$$

Rmk. If X, Y are CW complexes, there is an isomorphism

$$C^{cell}_*(X) \otimes C^{cell}_*(Y) \cong C^{cell}_*(X \times Y)$$

This gives a proof of Eilenberg-Zilber in this case.

Proof of Eilenberg-Zilber $\left[C_*(A) \otimes C_*(B) \xrightarrow[\nabla_{A,B}]{q\text{-}i} C_*(A \otimes B) \right]$

Strategy:
- define a map $\nabla_{A,B}$ natural in A, B
- show that $\nabla_{A,B}$ is a quasi-iso when $A = \mathbb{Z}\Delta[n]$, $B = \mathbb{Z}\Delta[m]$:

$$C_*(\Delta[n]) \otimes C_*(\Delta[m]) \longrightarrow C_*(\Delta[n] \times \Delta[m])$$

(easy since $\Delta[n], \Delta[m], \Delta[n] \times \Delta[m]$ are simplicially contractible, so homology is $\mathbb{Z}$ in degree 0)
- By Yoneda, every $A \in s\mathrm{Ab}$ is canonically a colimit of $\mathbb{Z}\Delta[n]$'s. The functors C_*, $\otimes$ preserve colimits (in each variable)

$\Rightarrow \nabla_{A,B}$ is a colimit of $\nabla_{\mathbb{Z}\Delta[n],\mathbb{Z}\Delta[m]}$

Problem: colimits do not preserve quasi-isomorphisms in general.

Solution 1: Use "homotopy colimits" instead of colimits. Homotopy colimits preserve quasi-isomorphisms by design. Moreover, every $A \in sAb$ is still canonically a homotopy colimit of $\mathbb{Z}\Delta[n]$'s, and C_*, $\otimes$ preserve homotopy colimits.

$\rightsquigarrow$ Use model categories or ∞-categories

Solution 2: upgrade quasi-iso to chain homotopy equivalence ("Acyclic models").

$A, B \in sAb$.

$$\nabla_{A,B} : C_*(A) \otimes C_*(B) \longrightarrow C_*(A \otimes B)$$

$$p+q=n,\ a \in A_p,\ b \in B_q,\quad \nabla(a \otimes b) = \sum_{\substack{\mu = \{\mu_1 < \cdots < \mu_p\} \\ \nu = \{\nu_1 < \cdots < \nu_q\} \\ \mu \cup \nu = \{0, \ldots, n-1\}}} \operatorname{sgn}(\mu,\nu)\ \underbrace{s_\nu}_{s_{\nu_q} \cdots s_{\nu_1}}(a) \otimes s_\mu(b)$$

(p,q)-shuffles

$\nabla_{A,B}$ is called the <u>shuffle map</u> or the <u>Eilenberg–Zilber map</u>.
It is straightforward to check that $\nabla_{A,B}$ is a chain map.

$$\Delta_{A,B} : C_*(A \otimes B) \longrightarrow C_*(A) \otimes C_*(B)$$

$$a \in A_n,\ b \in B_n \qquad \Delta(a \otimes b) = \sum_{p+q=n} d^p_{\text{front}}(a) \otimes d^q_{\text{back}}(b)$$

where
$$d^p_{\text{front}} = \Big(\underset{i \mapsto i}{[p] \hookrightarrow [n]}\Big)^*$$
$$d^q_{\text{back}} = \Big(\underset{i \mapsto i+p}{[q] \hookrightarrow [n]}\Big)^*$$

$\Delta_{A,B}$ is called the <u>Alexander–Whitney map</u>. Again, it is a chain map.

- $A = \mathbb{Z}\Delta[n]$, $B = \mathbb{Z}\Delta[m]$. Then $A \otimes B = \mathbb{Z}[\Delta[n] \times \Delta[m]]$

$$C_*(A) \otimes C_*(B) = C_*(\Delta[n]) \otimes C_*(\Delta[m])$$
$$C_*(A \otimes B) = C_*(\Delta[n] \times \Delta[m]).$$

We know H_* of both chain complexes is $\mathbb{Z}$ concentrated in degree 0

Moreover, ∇ and Δ induce isomorphism on H_0 (check on a generator)

$\Longrightarrow$ $\nabla_{\mathbb{Z}\Delta[n], \mathbb{Z}\Delta[m]}$ and $\Delta_{\mathbb{Z}\Delta[n], \mathbb{Z}\Delta[m]}$ are quasi-isomorphisms

<u>Proposition</u> Let $\mathcal{A}$ be an abelian category, $C_*, D_* \in Ch_{\geq 0}(\mathcal{A})$.
Suppose each C_i is projective and $H_i(D_*) = 0$ if $i \neq 0$.
Then the map

$$\mathrm{Hom}_{Ch(\mathcal{A})}(C_*, D_*)/_{\text{chain homotopy}} \xrightarrow{H_0} \mathrm{Hom}_{\mathcal{A}}(H_0(C_*), H_0(D_*))$$

is a bijection

<u>Proof</u>.

Surjectivity:

$$\begin{tikzcd}
 & C_2 \arrow[r] \arrow[d, dashed, "\exists"] & C_1 \arrow[r] \arrow[d, dashed, "\exists"] & C_0 \arrow[r] \arrow[d, dashed, "\exists"] & H_0(C_*) \arrow[r] \arrow[d] & 0 \\
\text{etc.} & D_2 \arrow[r] & D_1 \arrow[r] & D_0 \arrow[r, two heads] & H_0(D_*) \arrow[r] & 0
\end{tikzcd} \quad \text{(exact)}$$

Injectivity: $\varphi: C_* \to D_*$, $H_0(\varphi) = 0 \overset{?}{\Longrightarrow} \varphi$ is chain-homotopic to 0

$$\begin{tikzcd}
 & C_2 \arrow[r] \arrow[d, "\varphi_2"'] & C_1 \arrow[r] \arrow[d, "\varphi_1"] \arrow[dl, dashed, "\exists h_1"'] \arrow[d, shift right=4, "\varphi_1 - h_0 d"'] & C_0 \arrow[r] \arrow[d, "\varphi_0"] \arrow[dl, dashed, "\exists h_0"'] & H_0(C_*) \arrow[r] \arrow[d, "H_0(\varphi)=0"] & 0 \\
\text{etc.} & D_2 \arrow[r] & D_1 \arrow[r] & D_0 \arrow[r] & H_0(D_*) \arrow[r] & 0
\end{tikzcd} \quad \text{(exact)}$$

$\square$