Remark. Let $\mathrm{ProjRes}(\mathcal{A}) \subset \mathrm{Ch}_{\geq 0}(\mathcal{A})$ subcategory of P_* such that P_i is projective for all i and $H_i(P_*) = 0$ if $i \neq 0$.

The Proposition implies that

$$H_0 : \underset{\substack{\uparrow \\ \text{chain-homotopy} \\ \text{classes of morphisms}}}{h}\mathrm{ProjRes}(\mathcal{A}) \longrightarrow \mathcal{A}$$

is fully faithful.

$\mathcal{A}$ has <u>enough projectives</u> if every object is a quotient of a projective object (e.g. Mod_R). Then H_0 is essentially surjective, hence an equivalence. If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, the i-th <u>left derived functor</u> is:

$$L_i F : \mathcal{A} \xrightarrow{H_0^{-1}} h\mathrm{ProjRes}(\mathcal{A}) \subset h\mathrm{Ch}_{\geq 0}(\mathcal{A}) \xrightarrow{F} h\mathrm{Ch}_{\geq 0}(\mathcal{B}) \xrightarrow{H_i} \mathcal{B}$$

e.g. $\mathrm{Tor}_i^R(M,-) = L_i(M \otimes_R -) : \mathrm{Mod}_R \to \mathrm{Mod}_R$

$\mathrm{Ext}^i_R(-,M) = L_i \mathrm{Hom}_R(-,M) : \mathrm{Mod}_R \to \mathrm{Mod}_R^{op}$.

We apply the Proposition with $\mathcal{A} = \mathrm{Fun}(\Delta \times \Delta, \mathrm{Ab})$.

We have

$$\Phi_i : ([n],[m]) \longmapsto \bigoplus_{p+q=i} \left(C_p(\Delta[n]) \otimes C_q(\Delta[m]) \right)$$

$$\Psi_i : ([n],[m]) \longmapsto C_i(\Delta[n] \times \Delta[m])$$

$\Phi_*, \Psi_* \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$.

$H_i(\Phi_*) : ([n],[m]) \longmapsto H_i(C_*(\Delta[n]) \otimes C_*(\Delta[m])) = 0$ if $i \neq 0$.

$H_i(\Psi_*) : ([n],[m]) \longmapsto H_i(C_*(\Delta[n] \times \Delta[m])) = 0$ if $i \neq 0$.

<u>Claim</u> Φ_i, Ψ_i are projective objects in $\mathcal{A}$.

$$\Phi_i : ([n],[m]) \longmapsto \bigoplus_{p+q=i} C_p(\Delta[n]) \otimes C_q(\Delta[m]) \underset{\uparrow}{\cong} \bigoplus_{p+q=i} \mathbb{Z}[\mathrm{Hom}_{\Delta \times \Delta}(([p],[q]),([n],[m]))]$$

$$C_p(\Delta[n]) = \mathbb{Z}[\Delta[n]_p] = \mathbb{Z}[\mathrm{Hom}_\Delta([p],[n])]$$

$$\Phi_i = \bigoplus_{p+q=i} \mathbb{Z}[\mathrm{Hom}_{\Delta\times\Delta}(([p],[q]),-)] : \Delta\times\Delta \longrightarrow Ab$$

Similarly $\Psi_i = \mathbb{Z}[\mathrm{Hom}_{\Delta\times\Delta}(([i],[i]),-)] : \Delta\times\Delta \longrightarrow Ab.$

Now, if C is a small category and $x \in C$, then

$$\mathbb{Z}[\mathrm{Hom}_C(x,-)] : C \longrightarrow Ab$$

is projective in $\mathrm{Fun}(C, Ab)$. Indeed:

$$\mathrm{Nat}(\mathbb{Z}[\mathrm{Hom}_C(x,-)], F) \underset{\substack{\uparrow \\ \text{Yoneda}}}{\cong} F(x)$$

If $F \to G$ an epimorphism in $\mathrm{Fun}(C, Ab)$, then $F(y) \to G(y)$ is an epimorphism for all $y \in C$.

$\Rightarrow$ $\mathrm{Nat}(\mathbb{Z}[\mathrm{Hom}_C(x,-)], -)$ preserves epimorphisms, i.e. $\mathbb{Z}[\mathrm{Hom}_C(x,-)]$ is projective.

We have $\nabla : \Phi_* \longrightarrow \Psi_*$, $\Delta : \Psi_* \longrightarrow \Phi_*$ & both induce an isomorphism on H_0, inverse to one another.

$\underset{\text{Prop}}{\Rightarrow}$ $\Delta \circ \nabla \overset{S}{\simeq} \mathrm{id}$ via a chain-homotopy $S : \Phi_* \longrightarrow \Phi_{*+1}$

$\nabla \circ \Delta \overset{T}{\simeq} \mathrm{id}$ ———— $T : \Psi_* \longrightarrow \Psi_{*+1}$.

Recall: Yoneda lemma: if C is cocomplete, then

$$\mathrm{Fun}(\Delta, C) \overset{\simeq}{\longleftarrow} \mathrm{Fun}^L(sSet, C)$$

colimit-preserving functors

Additive version: if C is moreover additive, then

$$\mathrm{Fun}(\Delta, C) \overset{\simeq}{\longleftarrow} \mathrm{Fun}^L(sAb, C).$$

$$\Rightarrow A = \mathrm{Fun}(\Delta\times\Delta, Ab) \cong \mathrm{Fun}(\Delta, \mathrm{Fun}(\Delta, Ab)) \simeq \mathrm{Fun}^L(sAb, \mathrm{Fun}^L(sAb, Ab))$$

$$\cong \mathrm{Fun}^{L\times L}(sAb \times sAb, Ab)$$

preserves colimits in each variable

$\Rightarrow$ ∇, Δ, S, T extend uniquely to natural transformations between functors $sAb \times sAb \longrightarrow Ab$. $\square$

The cup product

Recall the Alexander–Whitney map:

$$AW : C_{p+q}(X\times Y) \longrightarrow C_p(X)\otimes C_q(Y)$$

$$\Big(\underset{[v_0,\dots,v_{p+q}]}{\Delta^{p+q}} \xrightarrow{\sigma} X\times Y\Big) \longmapsto \big(\pi_X\circ\sigma|_{[v_0,\dots,v_p]}\big)\otimes\big(\pi_Y\circ\sigma|_{[v_p,\dots,v_{p+q}]}\big)$$

Construction $X,Y\in \mathrm{Top}$, $A,B\in \mathrm{Ab}$

We construct a chain map

$$C^*(X,A)\otimes C^*(Y,B) \longrightarrow C^*(X\times Y, A\otimes B)$$

natural in X,Y,A,B, as follows:

$$\begin{aligned}
C^*(X,A)\otimes C^*(Y,B) &= \underline{\mathrm{Hom}}(C_*(X),A)\otimes \underline{\mathrm{Hom}}(C_*(Y),B)\\
&\big\downarrow \otimes\\
&\underline{\mathrm{Hom}}(C_*(X)\otimes C_*(Y), A\otimes B)\\
&\big\downarrow AW^*\\
&\underline{\mathrm{Hom}}(C_*(X\times Y), A\otimes B) = C^*(X\times Y, A\otimes B)
\end{aligned}$$

Explicitly: if $\varphi\in C^p(X,A)$, $\psi\in C^q(Y,B)$

$$\rightsquigarrow \big(\sigma:[v_0,\dots,v_{p+q}]\to X\times Y\big)\longmapsto \varphi\big(\pi_X\sigma|_{[v_0,\dots,v_p]}\big)\otimes\psi\big(\pi_Y\sigma|_{[v_p,\dots,v_{p+q}]}\big).$$

If R is a ring, we have $\mu: R\otimes R\to R$.

- The external product is

$$C^p(X,R)\otimes C^q(Y,R)\longrightarrow C^{p+q}(X\times Y, R\otimes R)\xrightarrow{\mu_*} C^{p+q}(X\times Y,R)$$

$$\varphi\otimes\psi\longmapsto \varphi\times\psi.$$

- When $X=Y$, the cup product is

$$C^p(X,R)\otimes C^q(X,R)\xrightarrow{\times} C^{p+q}(X\times X,R)\xrightarrow{\delta^*} C^{p+q}(X,R)$$

where $\delta: X\to X\times X$ is the diagonal.

This is denoted by $\varphi\otimes\psi \mapsto \varphi\cup\psi$ or $\varphi\psi$:

$$(\varphi\psi)(\sigma:[v_0,\dots,v_{p+q}]\to X) = \varphi(\sigma|_{[v_0,\dots,v_p]})\psi(\sigma|_{[v_p,\dots,v_{p+q}]}).$$

Since all these maps are chain maps, we get induced pairings:

$$H^p(X,A)\otimes H^q(Y,B) \longrightarrow H^{p+q}(X\times Y, A\times B)$$
$$H^p(X,R)\otimes H^q(Y,R) \xrightarrow{\times} H^{p+q}(X\times Y, R)$$
$$H^p(X,R)\otimes H^q(X,R) \xrightarrow{\cup} H^{p+q}(X,R)$$

__Lemma__ The cup product makes $C^*(X,R)$ and $H^*(X,R)$ into graded rings.

Pf: • Multiplicative unit is the constant cochain $1\in C^0(X,R) = \mathrm{Hom}(C_0(X),R)$

• Associativity: if $\varphi\in C^p$, $\psi\in C^q$, $\chi\in C^r$ then

$$\begin{matrix}\varphi(\psi\chi)\\ (\varphi\psi)\chi\end{matrix} : (\Delta^{p+q+r}\xrightarrow{\sigma} X)\mapsto \varphi(\sigma|_{[v_0,\dots,v_p]})\psi(\sigma|_{[v_p,\dots,v_{p+q}]})\chi(\sigma|_{[v_{p+q},\dots,v_{p+q+r}]}). \qquad \square$$

__Proposition__ Suppose R is a __commutative__ ring. Then the cohomology ring $H^*(X,R)$ is graded-commutative, i.e.:

$$\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha \qquad \big(|\alpha| = n \Leftrightarrow \alpha\in H^n(X,R)\big)$$

In particular, if α has odd degree, then $2\alpha^2 = 0$.

Pf: Pick representatives $\alpha\in C^p(X,R)$, $\beta\in C^q(X,R)$, $n=p+q$.

Let $\sigma:\Delta^n\longrightarrow X$, $\Delta^n = [v_0,\dots,v_n]$.

$$(\alpha\beta)(\sigma) = \alpha(\sigma|_{[v_0,\dots,v_p]})\beta(\sigma|_{[v_p,\dots,v_n]})$$
$$(\beta\alpha)(\sigma) = \alpha(\sigma|_{[v_q,\dots,v_n]})\beta(\sigma|_{[v_0,\dots,v_q]})$$

Define $\bar\sigma = \sigma|_{[v_n,\dots,v_0]}$ so $\bar\sigma|_{[v_0,\dots,v_p]} = \sigma|_{[v_n,\dots,v_q]}$

$$\rho_n: C_n(X)\longrightarrow C_n(X)$$
$$\sigma\mapsto \varepsilon_n\bar\sigma \quad \text{where } \varepsilon_n = (-1)^{n(n+1)/2}$$

We have: $\big((\alpha\circ\rho_p)(\beta\circ\rho_q)\big)(\sigma) = \varepsilon_p\varepsilon_q\ \alpha(\sigma|_{[v_p,\dots,v_0]})\,\beta(\sigma|_{[v_n,\dots,v_p]})$

$$\big((\beta\alpha)\circ\rho_n\big)(\sigma) = \varepsilon_n\ \alpha(\bar\sigma|_{[v_q,\dots,v_n]})\,\beta(\bar\sigma|_{[v_0,\dots,v_q]})$$

$$\varepsilon_n = (-1)^{pq}\varepsilon_p\varepsilon_q$$

<u>Claim</u>: ρ is a chain map and is chain-homotopic to id.

$\Rightarrow\quad [\alpha\beta] = [(\alpha\circ\rho_p)(\beta\circ\rho_q)] = (-1)^{pq}\,[(\beta\alpha)\circ\rho_n] = (-1)^{pq}\,[\beta\alpha]$

- ρ is a chain map: $\sigma : \Delta^n \to X$:

$$\rho\, d\sigma = \rho\Big(\sum_{i=0}^{n}(-1)^i\,\sigma|_{[v_0,\dots,\hat v_i,\dots,v_n]}\Big) = \varepsilon_{n-1}\sum_{i=0}^{n}(-1)^i\,\sigma|_{[v_n,\dots,\hat v_i,\dots,v_0]}$$

$$d\rho\sigma = \varepsilon_n\sum_{i=0}^{n}(-1)^i\,\bar\sigma|_{[v_0,\dots,\hat v_i,\dots,v_n]} = \varepsilon_n\sum_{i=0}^{n}(-1)^i\,\sigma|_{[v_n,\dots,\hat v_{n-i},\dots,v_0]}$$

These are equal, since $\varepsilon_n = (-1)^n\varepsilon_{n-1}$.

- ρ is chain homotopic to identity:

$\Delta^n\times I$: 

$\pi : \Delta^n\times I \longrightarrow \Delta^n$.

define $T : C_n(X) \longrightarrow C_{n+1}(X)$

$$T(\sigma) = \sum_{i=0}^{n}(-1)^i\,\varepsilon_{n-i}\,(\sigma\pi)|_{[v_0,\dots,v_i,w_n,\dots,w_i]}$$

$(n+1)$-simplex $\subset \Delta^n\times I$

Direct computation: $dT + Td = \rho - \mathrm{id}$. (see Hatcher Thm. 3.11) $\square$