Examples

- $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ with $|x| = 2$
- $H^*(\mathbb{R}P^n, \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$ with $|x| = 1$
- $H^*(S^n) \cong \Lambda_{\mathbb{Z}}(x) = \mathbb{Z}[x]/(x^2)$ exterior algebra on one generator x, $|x| = n$

  $\mathbb{Z} \oplus \mathbb{Z}$ $\qquad$ = square-zero extension of $\mathbb{Z}$ by $\mathbb{Z}$

- $H^*(\underbrace{S^{n_1} \times \dots \times S^{n_k}}_{n_i \text{ odd}}) \cong \Lambda_{\mathbb{Z}}(x_1, \dots, x_k)$ with $|x_i| = n_i$

  x_i anticommute

  $H^*(\underbrace{S^{n_1} \times \dots \times S^{n_k}}_{n_i \text{ even}}) \cong \mathbb{Z}[x_1, \dots, x_k]/(x_1^2, \dots, x_k^2)$ with $|x_i| = n_i$

  x_i commute

<u>Remark</u> (homology vs. cohomology)

Even though the groups $H^*(X)$ are determined by $H_*(X)$ up to isomorphism, Cohomology has two major advantages over homology:

1) $H^*(X)$ is a ring. This allows distinguishing more homotopy types, and is generally very useful for computations.

   Example: $\mathbb{R}P^3$ vs. $\mathbb{R}P^2 \vee S^3$ have same π_1, same H_*

   but: $H^*(\mathbb{R}P^3, \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^4)$ $\quad |x| = 1$.

   $$\not\cong$$

   $H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^3) \underset{\mathbb{Z}/2}{\times} \mathbb{Z}/2[y]/(y^2)$ $\quad |x| = 1, |y| = 3$

2) $H^*(-)$ is a contravariant functor (which is not determined by $H_*(-)$, since the splitting in the universal coeff. theorem is not natural).

   This is very useful, because many invariants of spaces are <u>representable</u> in hTop:

   - $\{\text{principal } G\text{-bundles on } X\} \cong [X, BG]$
   - $\underbrace{\mathrm{Vect}_n(X)}_{\text{real vector bundles of rank } n} \cong [X, BGL_n(\mathbb{R})]$
   - $H^n(X, A) \cong [X, K(A,n)]$

     Eilenberg-MacLane space $\pi_i K(A,n) = \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$
   - etc.

   Cohomology classes of these universal spaces $BGL_n(\mathbb{R})$, BG, $K(A,n)$ can be pulled back to X. This is the theory of <u>characteristic classes</u>:

e.g. $H^*(BGL_n(\mathbb{R}), \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \dots, w_n]$ $\quad |w_i| = i$

$\rightsquigarrow$ $w_i : \mathrm{Vect}_n(X) \longrightarrow H^i(X, \mathbb{Z}/2)$ (Stiefel–Whitney classes)

$H^*(K(A,n), B)$ $\rightsquigarrow$ Steenrod operations, which act on the cohomology of any space.

Remark (Homology as a coalgebra)

Homology still has an external product:

$$C_*(X,A) \otimes C_*(Y,B) \xrightarrow{E\text{-}Z} C_*(X \times Y, A \times B).$$

$$H_p(X,A) \otimes H_q(Y,B) \longrightarrow H_{p+q}(X \times Y, A \times B)$$

So if $X = Y$, R a ring, we get:

$$\bigoplus_{p+q=n} H_p(X,R) \otimes H_q(X,R) \xrightarrow[\substack{\text{if } R \text{ is} \\ \text{a field}}]{\cong} H_n(X \times X, R) \xleftarrow{\delta_*} H_n(X,R)$$

$\rightsquigarrow$ graded coalgebra structure on $H_*(X,R)$ when R is a field.

At the level of chains, however,

$C_*(X)$ is a coalgebra with comultiplication $AW \circ \delta_*$

$C^*(X)$ is an algebra with multiplication $\delta^* \circ AW^*$

Poincaré duality

- Cohomology with compact support.

  A cochain $\varphi : C_n(X) \to A$ has compact support if $\exists\, K \subset X$ compact such that $\varphi|_{C_n(X \setminus K)} = 0$

  $\rightsquigarrow$ $C^*_c(X,A) \subset C^*(X,A)$ subcomplex. , $H^*_c(X,A)$

- Borel–Moore homology. A locally finite chain on X with coeff. in A is a formal sum $\sum_{\sigma : \Delta^n \to X} a_\sigma \sigma$, $a_\sigma \in A$

  such that for every $K \subset X$ compact, $\{\sigma \mid \sigma(\Delta^n) \cap K \neq \emptyset \text{ and } a_\sigma \neq 0\}$ is finite.

  $\rightsquigarrow$ $C^{BM}_*(X,A) \supset C_*(X,A)$, $H^{BM}_*(X,A)$.

- Twisted coefficients.

Def. A <u>local system</u> of abelian groups on a space X is a functor
$$A : \Pi_1(X) \longrightarrow Ab$$

We can define:
$$\begin{array}{c} C_n(X,A) = \bigoplus_{\sigma:\Delta^n\to X} A(\sigma(v_0)) \\ \big\downarrow d \\ C_{n-1}(X,A) \end{array}$$

$$d(a\sigma) = \sigma(v_0 \to v_1)_*(a)\, d_0\sigma + \sum_{i=1}^{n} (-1)^i\, a\, d_i\sigma .$$

$$C^n(X,A) = \prod_{\sigma:\Delta^n\to X} A(\sigma(v_0)) ,$$ d similarly defined.

$\rightsquigarrow$ $H_*(X,A)$, $H^*(X,A)$, $H^*_c(X,A)$, $H^{BM}_*(X,A)$.

<u>Example:</u> $\Pi_1(\mathbb{RP}^2) \simeq B\mathbb{Z}/2$

Let $\tilde{\mathbb{Z}}$ be the functor $B\mathbb{Z}/2 \longrightarrow Ab$ (sign representation)
$$\begin{array}{ccc} B\mathbb{Z}/2 & \longrightarrow & Ab \\ * \circlearrowleft & \longmapsto & \mathbb{Z} \circlearrowleft \cdot(-1) \end{array}$$

$\rightsquigarrow$ $H_*(\mathbb{RP}^2,\mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \\ \mathbb{Z}/2 & 1 \\ 0 & 2 \end{cases}$ but $H_*(\mathbb{RP}^2,\tilde{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2 & 0 \\ 0 & 1 \\ \mathbb{Z} & 2 \end{cases}$

$H^*(\mathbb{RP}^2,\mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \\ 0 & 1 \\ \mathbb{Z}/2 & 2 \end{cases}$ $\qquad H^*(\mathbb{RP}^2,\tilde{\mathbb{Z}}) = \begin{cases} 0 & 0 \\ \mathbb{Z}/2 & 1 \\ \mathbb{Z} & 2 . \end{cases}$

If M is a (topological) manifold, there is a canonical degree 2 covering $\tilde{M} \to M$ called the <u>orientation covering</u> (M orientable $\Leftrightarrow$ $\tilde{M} \to M$ is trivial)
(e.g.: $S^2 \to \mathbb{RP}^2$)

This corresponds to a functor
$$\begin{tikzcd} \Pi_1(M) \arrow[r] \arrow[dr, dashed] \arrow[ddr, bend right, "\tilde{\mathbb{Z}}"'] & Set \\ & \{2\text{-pt sets}\}^{\simeq} \arrow[u, phantom, "\cup"] \arrow[d, "\{a,b\} \mapsto \langle a,b \mid ab \rangle"] \\ & Ab \end{tikzcd}$$

$\tilde{A} := A \otimes \tilde{\mathbb{Z}}$

e.g.: $\widetilde{\mathbb{Z}/2} = \mathbb{Z}/2$

$\tilde{\mathbb{Z}} \otimes \tilde{\mathbb{Z}} \simeq \mathbb{Z}$

<u>Theorem</u> (Poincaré duality)

Let M be a manifold of dim n (without boundary), and A a local system of abelian groups on M.

Then there are quasi-isomorphisms

$$\begin{array}{ccc} C^*_c(M,A)[n] & \xrightarrow{\sim} & C_*(M,\tilde{A}) \\ \cap & & \cap \\ C^*(M,A)[n] & \xrightarrow{\sim} & C^{BM}_*(M,\tilde{A}) \end{array}$$

$$\Rightarrow \quad H^{n-i}_c(M,A) \cong H_i(M,\tilde{A})$$
$$H^{n-i}(M,A) \cong H^{BM}_i(M,\tilde{A})$$

<u>Corollary:</u> For a field F, $H^i(M,F) \cong H_i(M,F)^*$

$$\Rightarrow \quad H^i(M,F) \cong H^{n-i}_c(M,\tilde{F})^* .$$

<u>Def.</u> • R ring, M is <u>R-orientable</u> if $\tilde{R} \overset{\exists}{\cong} R$ R-linear isomorphism.

An <u>R-orientation</u> is a choice of such isomorphism

(there are $R^\times$ many choices, or none)

• orientable $\equiv$ $\mathbb{Z}$-orientable.

→ if M is R-oriented & A is a local system of R-modules, then

$$H_i(M,A) \cong H^{n-i}_c(M,A)$$

→ if M is compact: $H_i(M,\tilde{A}) \cong H^{n-i}(M,A)$.

• Every manifold is uniquely $\mathbb{Z}/2$-orientable since $\widetilde{\mathbb{Z}/2} = \mathbb{Z}/2$.

• $\mathbb{C}P^n$ is $\mathbb{Z}$-orientable (more generally, any complex manifold is $\mathbb{Z}$-orientable)

• $\mathbb{R}P^n$ is $\mathbb{Z}$-orientable $\Leftrightarrow$ n is odd.