Why homotopy?

- Most invariants in algebraic topology are homotopy invariant.
- Classifying topological spaces up to homeomorphisms is far too difficult, except in some very special cases (e.g. surfaces).
- Spaces up to homotopy equivalence are much more accessible.
  For example, if $f: X \longrightarrow Y$ is a continuous between "nice enough" spaces (e.g. smooth manifolds), then f is a homotopy equivalence if and only if $\pi_n(f)$ is an isomorphism for all $n \geq 0$.

Example We will define functors $H_n : \mathrm{Top} \longrightarrow \mathrm{Ab}$, $n \geq 0$ such that:

1) H_n is homotopy invariant
2) $$H_n(S^m) \cong \begin{cases} \mathbb{Z} & \text{if } n=0 \text{ or } m \\ 0 & \text{otherwise.} \end{cases} \quad (\text{if } m \geq 1), \quad H_n(S^0) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } n=0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1) if $n \neq m$, $S^n \not\simeq S^m$

2) if $n \neq m$, $\mathbb{R}^n \not\cong \mathbb{R}^m$.

Pf. 1) $H_n(S^n) \cong \mathbb{Z} \not\cong 0 = H_n(S^m) \overset{H_n \text{ homotopy invariant}}{\Longrightarrow} S^n \not\simeq S^m$.

2) Suppose $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a homeo.

$$\begin{array}{ccc} \mathbb{R}^n \setminus \{0\} & \cong & \mathbb{R}^m \setminus \{f(0)\} \\ \wr\wr & & \wr\wr \\ S^{n-1} & & S^{m-1} \end{array} \Longrightarrow S^{n-1} \simeq S^{m-1}$$
contradicts 1). $\square$.

Example (surfaces)

A closed surface is a compact Hausdorff space in which every point has an open neighborhood homeomorphic to $\mathbb{R}^2$.

- $g \geq 0$, Σ_g = "torus with g holes"

$\Sigma_0 = S^2$ $\quad$ $\Sigma_1 = S^1 \times S^1$ $\quad$ Σ_2 $\quad$ ... $\quad$ (Riemann surfaces)

- for $h \geq 1$, $N_h = (\mathbb{R}P^2)^{\# h}$ where # is connected sum:

N_2 = Klein bottle.

Theorem (classification of surfaces) Every connected closed surface is homeomorphic to exactly one of these.

We will see: $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$
$H_1(N_h) \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2$ $\qquad (\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z})$

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§ 2. The fundamental group(oid)

Definition A groupoid is a category in which every morphism is an isomorphism. We denote by Gpd the category whose objects are groupoids and whose morphisms are functors.

Examples

- If C is a category, there is a largest subcategory $C^{iso} \subset C$ which is a groupoid.
- If G is a group, we denote by BG the groupoid with a single object $*$ and $\mathrm{Aut}_{BG}(*) = G$, $g \circ h = gh$. $\quad * \circlearrowleft G$
- If G is a group acting on a set X (on the left), the action groupoid $X/\!/G$ as follows:
  - $\mathrm{Ob}(X/\!/G) = X$
  - $\mathrm{Hom}_{X/\!/G}(x,y) = \{ g \in G \mid x = gy \}$
  - composition:
$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend right, "gh"'] & y \arrow[r, "h"] & z \end{tikzcd} \qquad x = gy \quad y = hz \Rightarrow x = (gh)z$$

Remark: $BG = */\!/G$.

Proposition (classification of groupoids up to equivalence)
Let Γ be a groupoid and let E be a set of representatives of isomorphism classes of objects of Γ. Then there is an equivalence

$$\Gamma \simeq \coprod_{x \in E} B\mathrm{Aut}_\Gamma(x) \qquad \left(\begin{tikzcd} E \arrow[d, phantom, "\cap"] \arrow[dr, "\cong"] & \\ \mathrm{Ob}(\Gamma) \arrow[r, two heads] & \mathrm{Ob}(\Gamma)/\cong \end{tikzcd} \right)$$

Proof. Note that $\coprod_{x \in E} B\mathrm{Aut}_\Gamma(x)$ is a subcategory of Γ with objects E & morphisms endomorphisms.

The inclusion functor $\coprod_{x\in E} B\mathrm{Aut}_\Gamma(x) \hookrightarrow \Gamma$ is fully faithful because:

$$x,y \in E: \qquad \mathrm{Hom}_\Gamma(x,y) = \begin{cases} \emptyset & \text{if } x \neq y \\ \mathrm{Aut}_\Gamma(x) & \text{if } x = y. \end{cases}$$

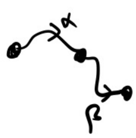
It is also essentially surjective since every $x \in \Gamma$ is isomorphic to some $y \in E$. $\square$.

Example: • In $X/\!/G$: $\left(\mathrm{Ob}(X/\!/G)/\cong\right) \cong X/G$ set of orbits
$\mathrm{Aut}_{X/\!/G}(x) = \mathrm{Stab}_G(x)$ stabilizer group.

$$\Rightarrow \quad X/\!\!/G \simeq \coprod_{[x]\in X/G} B\mathrm{Stab}_G(x).$$

- If $\mathrm{Vect}(k)$ is the category of finite-dim. vector spaces over a field k, then
$$\mathrm{Vect}(k)^{iso} \simeq \coprod_{n\geq 0} BGL_n(k)$$
$GL_n(k)$ = invertible $n\times n$ matrices with coeff. in k.

<u>Definition</u> $X \in \mathrm{Top}$, let $\alpha, \beta : I \longrightarrow X$ paths in X with $\alpha(1)=\beta(0)$.



- The <u>concatenation</u> of α and β is
$$\alpha * \beta : I \longrightarrow X, \quad (\alpha*\beta)(s) = \begin{cases} \alpha(2s) & \text{if } s\in[0,\frac{1}{2}] \\ \beta(2s-1) & \text{if } s \in [\frac{1}{2},1]. \end{cases}$$
- The <u>reverse</u> of α is $\bar{\alpha} : I \longrightarrow X$, $\bar{\alpha}(s) = \alpha(1-s)$
- The <u>constant path</u> at $x \in X$ is $c_x : I \to X$, $c_x(s) = x$.

<u>Proposition</u>. (Recall: $\simeq_p$ path-homotopy $\Longleftrightarrow$ homotopy rel. $\{0,1\}$.)

1)
- if $\alpha \simeq_p \alpha'$ then $\alpha * \beta \simeq_p \alpha' * \beta$
- if $\beta \simeq_p \beta'$ then $\alpha * \beta \simeq_p \alpha * \beta'$.

2) if $\alpha(1)=\beta(0)$ and $\beta(1)=\gamma(0)$, then
$$\alpha * (\beta * \gamma) \simeq_p (\alpha*\beta)*\gamma.$$

3)
- $\alpha * c_{\alpha(1)} \simeq_p \alpha$
- $c_{\alpha(0)} * \alpha \simeq_p \alpha$

4)
- $\alpha * \bar{\alpha} \simeq_p c_{\alpha(0)}$
- $\bar{\alpha} * \alpha \simeq_p c_{\alpha(1)}$

<u>Corollary / Definition</u> There is a groupoid $\Pi_1(X)$ with:

- $\mathrm{Ob}(\Pi_1(X)) = X$
- $\mathrm{Hom}_{\Pi_1(X)}(x,y) = \left\{ \alpha : I \longrightarrow X \text{ continuous} \;\middle|\; \begin{array}{l}\alpha(0)=x \\ \alpha(1)=y\end{array} \right\} / \simeq_p$
- $[\alpha]_p \in \mathrm{Hom}(x,y)$, $[\beta]_p \in \mathrm{Hom}(y,z)$, then
$$[\beta]_p \circ [\alpha]_p = [\alpha * \beta]_p.$$

$\Pi_1(X)$ is called the <u>fundamental groupoid</u> of X.

Proof of proposition

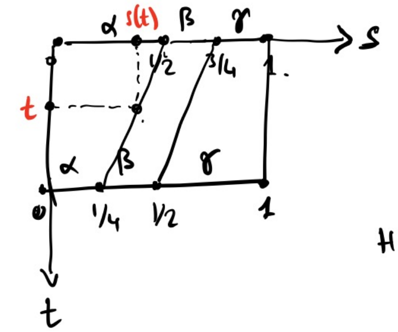
1) Let $H: I\times I \longrightarrow X$ be a path-homotopy from α to α'.

Then $$I\times I \longrightarrow X, \quad (s,t) \longmapsto \begin{cases} H(2s,t) & \text{if } s\in[0,1/2] \\ \beta(2s-1) & \text{if } s\in[1/2,1] \end{cases}$$

is a path-homotopy from $\alpha*\beta$ to $\alpha'*\beta$.

2) $\alpha*(\beta*\gamma)$

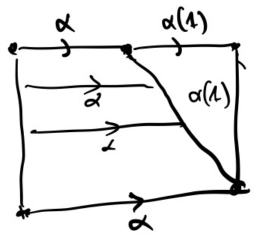
$(\alpha*\beta)*\gamma$



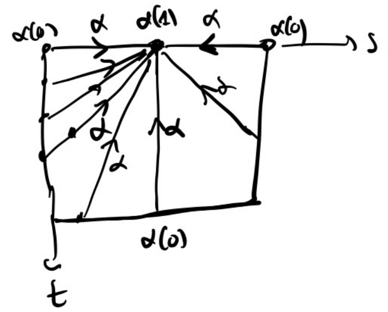
$H: I\times I \longrightarrow X$

$$H(s,t) = \begin{cases} \alpha\left(\frac{s}{s(t)}\right) & \text{if } s\in[0,s(t)] \\ \beta\left(4(s-s(t))\right) & \text{if } s\in[s(t),s(t)+\frac{1}{4}] \\ \gamma\left(\frac{s-s(t)-\frac{1}{4}}{s(1-t)}\right) & \text{if } s\in[s(t)+\frac{1}{4},1] \end{cases}$$

3) $\alpha * c_{\alpha(1)} \simeq_p \alpha$



4) $\alpha * \overline{\alpha} \simeq_p c_{\alpha(0)}$



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