Functoriality in X: if $f : X \to Y$ is a continuous map, define

$$\Pi_1(f) = f_* : \Pi_1(X) \longrightarrow \Pi_1(Y)$$

$$(x \in X) \longmapsto f(x) \in Y$$

$$([\alpha]_p : x \to y) \longmapsto [f\circ\alpha]_p : f(x) \to f(y)$$

This is well-defined, because if $\alpha \overset{H}{\simeq}_p \alpha'$ then $f\circ\alpha \overset{f\circ H}{\simeq}_p f\circ\alpha'$.

Easy to check:
- $\Pi_1(\mathrm{id}_X) = \mathrm{id}_{\Pi_1(X)}$
- $\Pi_1(g\circ f) = \Pi_1(g)\circ\Pi_1(f)$.

$\Rightarrow$ The fundamental groupoid is a functor

$$\Pi_1 : \mathrm{Top} \longrightarrow \mathrm{Gpd}.$$

Lemma The functor Π_1 preserves arbitrary products: if $(X_i)_{i\in I}$ is a family of top. spaces, then $\Pi_1\left(\prod_{i\in I} X_i\right) \xrightarrow{\cong} \prod_{i\in I} \Pi_1(X_i)$.

Proof. This follows from the universal property of products. $\square$

Proposition (Homotopy invariance of Π_1)

1) If $f \simeq g : X \to Y$, then $\Pi_1(f) \overset{\text{natural isomorphism of functors}}{\cong} \Pi_1(g) : \Pi_1(X) \to \Pi_1(Y)$.

2) The functor $\Pi_1 : \mathrm{Top} \to \mathrm{Gpd}$ takes homotopy equivalences to equivalences of groupoids.

3) If X is contractible, then $\Pi_1(X) \underset{\text{equiv. of categories}}{\simeq} *$

Proof.

1) Let $H : X \times I \to Y$ be a homotopy from f to g.

$$\begin{tikzcd}
\Pi_1(X\times I) \arrow[r, "\Pi_1(H)"] \arrow[d, phantom, "\cong"', "\text{Lemma}"] & \Pi_1(Y) \\
\Pi_1(X)\times\Pi_1(I) &
\end{tikzcd}$$

In $\Pi_1(I)$, there is an isomorphism $[\alpha]_p : 0 \xrightarrow{\cong} 1$ (e.g. $\alpha = \mathrm{id}_I$)

$$\begin{tikzcd}
\Rightarrow \Pi_1(H)(-,0) \arrow[r, "\cong"] \arrow[d, phantom, "\parallel"] & \Pi_1(H)(-,1) \arrow[d, phantom, "\parallel"] : \Pi_1(X) \to \Pi_1(Y) \\
\Pi_1(f) & \Pi_1(g)
\end{tikzcd}$$

2) $f : X \to Y$, $g : Y \to X$, $f\circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. Apply Π_1 and use 1).

$\Rightarrow$ $\Pi_1(f)$ and $\Pi_1(g)$ are quasi-inverse to one another.

3) Apply 2) to $X \longrightarrow *$. $\square$

Definition. Let $(X, x_0) \in \mathrm{Top}_*$. The *fundamental group* of X at x_0 is the group

$$\pi_1(X, x_0) = \mathrm{Aut}_{\Pi_1(X)}(x_0).$$

This defines a functor $\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$.

Explicitly:
- elements are path-homotopy classes of *loops* at x_0.
- group operation is concatenation.

Proposition (Homotopy invariance of π_1)

If $f : X \longrightarrow Y$ is a homotopy equivalence and $x_0 \in X$, then

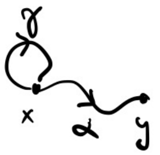
$$\pi_1(f) = f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof: $f_* : \Pi_1(X) \longrightarrow \Pi_1(Y)$ is an equivalence. In particular, it is fully faithful. $\square$

Dependence on the base point

If α is a path from x to y in X, there is an isomorphism



$$\alpha_* : \pi_1(X, x) \xrightarrow{\cong} \pi_1(X, y)$$
$$[\gamma]_p \longmapsto [\bar{\alpha} * \gamma * \alpha]_p$$

with inverse $\bar{\alpha}_*$.

This defines a functor $\pi_1(X, -) : \Pi_1(X) \longrightarrow \mathrm{Grp}$.

(special case of $\mathrm{Aut}_\Gamma(-) : \Gamma \to \mathrm{Grp}$.)

Warning: In general, different paths give different isomorphisms: if α, β are paths from x to y, then

$$\alpha_*([\gamma]_p) = g^{-1} \beta_*([\gamma]_p)\, g \qquad \text{where } g = [\bar{\beta} * \alpha]_p \in \pi_1(X, y).$$

So if $\pi_1(X, y)$ is not abelian, there is no canonical isomorphism $\pi_1(X, x) \cong \pi_1(X, y)$.

Goal: develop techniques to compute π_1. For example:

- $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(S^n) = 0$ if $n \geqslant 2$.
- $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ for $n \geqslant 2$
- $\pi_1(\infty) \cong \mathbb{Z} * \mathbb{Z}$ (free group on 2 generators) $= \langle a, b \rangle$
- $\pi_1($  $) \cong \mathbb{Z} \oplus \mathbb{Z}$ (free abelian group on 2 generators) $= \langle a, b \mid aba^{-1}b^{-1} \rangle$
- $\pi_1($Klein bottle$) \cong \langle a, b \mid abab^{-1} \rangle$

Higher homotopy groups

$$\pi_1(X, x_0) = [\overset{\overset{S^1}{\cong}}{I/\{0,1\}}, (X, x_0)]_*$$

if $A \subset X$, X/A is the pushout

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/A \end{tikzcd}$$

Let ∂I^n be the boundary of the n-cube I^n, so

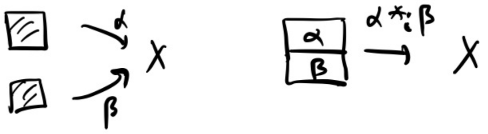
$$I^n/\partial I^n \cong D^n/\partial D^n \cong S^n .$$

If $(X, x_0) \in Top_*$, we define

$$\pi_n(X, x_0) = [I^n/\partial I^n, (X, x_0)]_*$$

Group structure: for each $i \in \{1, \dots, n\}$, we can concatenate in the i-direction:

$n = 2$:



$$\alpha, \beta : I^n/\partial I^n \longrightarrow (X, x_0)$$

$$(\alpha *_i \beta)(s_1, \dots, s_n) = \begin{cases} \alpha(s_1, \dots, 2s_i, \dots, s_n) & \text{if } s_i \in [0, \frac{1}{2}] \\ \beta(s_1, \dots, 2s_i - 1, \dots, s_n) & \text{if } s_i \in [\frac{1}{2}, 1]. \end{cases}$$

Then $*_i$ induces a group structure on $\pi_n(X, x_0)$.

Moreover, this group structure does not depend on i and it is abelian if $n \geqslant 2$. (exercise)

Remarks 1) for any $n \geqslant 0$, we have

path-connected components.

$$\pi_n(X, x_0) = \pi_0 \, \Omega^n_{x_0}(X)$$

where $\Omega_{x_0}(X) \subset X^I$ subspace of loops at x_0.

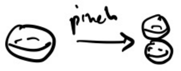
So $\Omega^n_{x_0}(X) \subset X^{I^n}$ is the subspace of $I^n \to X$ that send ∂I^n to x_0.

2) The functors π_n are represented by S^n in $hTop_*$:

$$\pi_n(X,x_0) = [S^n, (X,x_0)]_*$$

The fact that π_n is group-valued is equivalent, by the Yoneda lemma, to the fact that S^n is a cogroup object in $hTop_*$.

$n=1$: 

$n=2$: 

. . .

$S^1 \longrightarrow S^1 \vee S^1$ $\qquad$ $S^2 \longrightarrow S^2 \vee S^2$

3) We have a functor $\pi_n(X,-) : \Pi_1(X) \longrightarrow \begin{cases} Set_* & \text{if } n=0 \\ Grp & \text{if } n=1 \\ Ab & \text{if } n\geqslant 2. \end{cases}$

4) We have $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geqslant 1$,

and $\pi_m(S^n) = 0$ for $0 < m < n$.

For $n=1$, we also have $\pi_m(S^1) = 0$ for $m > 1$.

However, $\pi_m(S^n) \neq 0$ in general for $m > n$.

For example, $\pi_3(S^2) \cong \mathbb{Z}$, generated by the Hopf map
$$\begin{tikzcd} S^3 \arrow[r, "\eta"] \arrow[d, phantom, "\cap" rotate=90] & S^2 \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{C}^2 \setminus \{0\} \arrow[r, two heads] & \mathbb{C}P^1 \end{tikzcd}.$$

It was proved in 2015 that $\pi_m(S^2) \neq 0$ for <u>all</u> $m \geqslant 2$.