§3. The Seifert-van Kampen theorem

Theorem (Seifert-van Kampen, groupoid version)
Let X be a topological space, $U, V \subset X$ subspaces such that $X = \overset{\text{interior of } U}{U^\circ} \cup V^\circ$.
Then the square of groupoids

$$\begin{tikzcd} \Pi_1(U \cap V) \arrow[r] \arrow[d] & \Pi_1(U) \arrow[d] \\ \Pi_1(V) \arrow[r] & \Pi_1(X) \end{tikzcd}$$

is a pushout (in the category Gpd).

Theorem (Seifert-van Kampen, group version)
Let $X \in \mathrm{Top}$, $U, V \subset X$ with $X = U^\circ \cup V^\circ$. Suppose that $U \cap V$ is path-connected.
Then, for any $x \in U \cap V$, the square of groups

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\ \pi_1(V, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

is a pushout in Grp.

Theorem If $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, then $\pi_1(S^1, 1) \cong \mathbb{Z}$.

$$[s \mapsto e^{2\pi i s}]_p \longleftarrow 1$$

Lemma (Lebesgue lemma)
Let (X, d) be a compact metric space and $(U_i)_{i \in I}$ an open cover of X.
Then there exists $\varepsilon > 0$ such that: for every $x \in X$, the ball $B(x, \varepsilon)$ is contained in U_i for some i.

Proof. $\forall x \in X$, $\exists\, i_x \in I$, and $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset U_{i_x}$.
Since X is compact $\exists\, Y \subset X$ finite such that $X = \bigcup_{y \in Y} B(y, \varepsilon_y/2)$

Let $\varepsilon = \min_{y \in Y} \varepsilon_y/2$. This ε works:

If $x \in X$, there exists $y \in Y$ with $x \in B(y, \varepsilon_y/2)$.
Hence $B(x, \varepsilon) \subset B(x, \varepsilon_y/2) \underset{\uparrow\, \triangle \text{ inequality.}}{\subset} B(y, \varepsilon_y) \subset U_{i_y}$. $\square$

Proof of Seifert–van Kampen.

We will prove that $\Pi_1(X)$ satisfies the universal property of the pushout:

$$\begin{tikzcd}
\Pi_1(U\cap V) \arrow[r] \arrow[d] & \Pi_1(U) \arrow[d] \arrow[dr, bend left, "f"] & \\
\Pi_1(V) \arrow[r] \arrow[rr, bend right, "g"'] & \Pi_1(X) \arrow[r, dashed, "\exists! h"] & \Gamma
\end{tikzcd}$$

On objects:

$$\begin{tikzcd}
U\cap V \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout in Set}"] & U \arrow[d] \arrow[dr, "f"] & \\
V \arrow[r] \arrow[rr, bend right, "g"'] & X \arrow[r, dashed, "\exists! h"] & Ob(\Gamma).
\end{tikzcd}$$

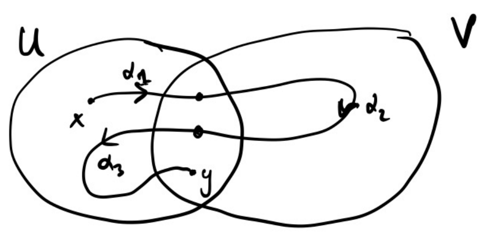
$$h(x) = \begin{cases} f(x) & \text{if } x\in U \\ g(x) & \text{if } x \in V. \end{cases}$$

On morphisms

Let $\alpha : I \longrightarrow X$ be a path from x to y. Since $X = U^\circ \cup V^\circ$,
$I = \alpha^{-1}(U^\circ) \cup \alpha^{-1}(V^\circ)$.

By the Lebesgue lemma, there exists $\varepsilon > 0$ s.t. every ball of radius ε in I is contained in either $\alpha^{-1}(U^\circ)$ or $\alpha^{-1}(V^\circ)$.

Hence, if $n > \frac{1}{2\varepsilon}$ then $\alpha\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) \subset U$ or V.



Let α_k be the path $I \cong \left[\frac{k-1}{n}, \frac{k}{n}\right] \xrightarrow{\alpha|_{..}} X$.
$s \mapsto \frac{s+k-1}{n}$

Then $\alpha \simeq_p \alpha_1 * \alpha_2 * \cdots * \alpha_n$.

We define $\tilde{h}(\alpha, n) : h(x) \longrightarrow h(y)$ in Γ by

$$\tilde{h}(\alpha, n) = f_n([\alpha_n]_p) \circ \cdots \circ f_1([\alpha_1]_p)$$

where $f_k = \begin{cases} f & \text{if } \mathrm{Im}(\alpha_k) \subset U \\ g & \text{if } \mathrm{Im}(\alpha_k) \subset V \end{cases}$

If $\mathrm{Im}(\alpha_k) \subset U \cap V$, then $f([\alpha_k]_p) = g([\alpha_k]_p)$, so this makes sense.

If $h : \Pi_1(X) \to \Gamma$ exists, then we must have $h([\alpha]_p) = \tilde{h}(\alpha, n)$ for $n \gg 0$, so h is unique.

<u>Claim</u> $\tilde{h}(\alpha, n)$ depends only on $[\alpha]_p$.

Assuming this claim, we can define $h : \Pi_1(X) \to \Gamma$ by $h([\alpha]_p) = \tilde{h}(\alpha, n)$ for $n \gg 0$.

This is a functor:

- $h([c_x]) = \tilde{h}(c_x, 1) = (f \text{ or } g)([c_x]_p) = \mathrm{id}_{h(x)}$
- $h([\beta]_p) \circ h([\alpha]_p) = \tilde{h}(\beta, n) \circ \tilde{h}(\alpha, n) = \tilde{h}(\alpha * \beta, 2n) = h([\alpha * \beta]_p)$

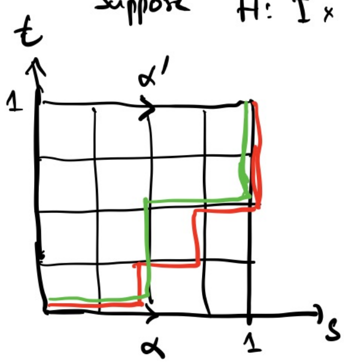
Also, it is clear that $h|_{\Pi_1(U)} = f$ and $h|_{\Pi_1(V)} = g$.

<u>Proof of Claim</u>:

It is clear that if $n \mid n'$ then $\tilde{h}(\alpha, n) = \tilde{h}(\alpha, n')$.

$\Rightarrow$ $\tilde{h}(\alpha, n)$ does not depend on n.

Suppose $H : I \times I \to X$ is a path-homotopy from α to α'.



By Lebesgue's lemma, applied to $I \times I$, there exists n such that $H([\frac{k-1}{n}, \frac{k}{n}] \times [\frac{\ell-1}{n}, \frac{\ell}{n}]) \subset U$ or V.

We want to show that $\tilde{h}(\alpha, n) = \tilde{h}(\alpha', n)$.

<u>Observation</u>: if $\sigma : I \times I \to X$, then $\sigma(-, 0) * \sigma(1, -) \simeq_p \sigma(0, -) * \sigma(-, 1)$.

(for example using a straight-line homotopy)

$\tilde{h}(\alpha, n) = \tilde{h}(\alpha * c_{\alpha(1)}, 2n) = \cdots$ by the observation.

$\cdots = \tilde{h}($  $, 2n) = \tilde{h}($  $, 2n) = \cdots$

$\cdots = \tilde{h}(c_{\alpha(0)} * \alpha', 2n) = \tilde{h}(\alpha', n)$ $\square$

Pushout of groups

Consider a diagram of groups

$$\begin{tikzcd} G_0 \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & G_1 \\ G_2 & \end{tikzcd}$$

The pushout is

$$G_1 \underset{G_0}{*} G_2 = \left\{ (g_1, g_2, \dots g_n) \;\middle|\; \begin{array}{l} n \geq 0 \\ g_i \in G_1 \sqcup G_2 \end{array} \right\} \Big/$$

disjoint union of sets

equiv relation generated by:

- $(\dots, g, g', \dots) \sim (\dots, g'', \dots)$ if $g'' = gg'$ in G_1 or G_2
- $(\dots, e, \dots) \sim (\dots, \dots)$
- $(\dots, g_1, \dots) \sim (\dots, g_2, \dots)$ if $\exists\, g_0 \in G_0$ such that $g_1 = \varphi_1(g_0)$, $g_2 = \varphi_2(g_0)$

with group structure given by concatenation of sequences.

Examples.

- If $G_2 = \{e\}$, then $G_1 \underset{G_0}{*} \{e\} \cong G_1 /$ normal subgroup generated by $\varphi_1(G_0)$.
- $\mathbb{Z} * \mathbb{Z}$ is the free group on 2 generators (it is not abelian)
- $\mathbb{Z}/2 * \mathbb{Z}/3 \cong PSL_2(\mathbb{Z}) \cong SL_2(\mathbb{Z}) / \{\pm I\}$ modular group.