Digression: presenting groups by generators and relations.

Let E be a set. The free group on E is

$$\langle E\rangle = \{ e_1 e_2 \cdots e_n \mid n \geq 0,\ e_i \in E \amalg \underset{\{e^{-1} \mid e\in E\}}{E^{-1}} \} \Big/ \begin{matrix} wee^{-1}w' \sim ww' \\ we^{-1}ew' \sim ww' \end{matrix}$$

with group operation given by concatenation.

Universal property: $\mathrm{Hom}_{Grp}(\langle E\rangle, G) \xrightarrow{\cong} \mathrm{Hom}_{Set}(E, G)$, ie:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & G \\ \langle E\rangle \arrow[ur, dashed, "\exists!\text{ group homomorphism } \hat{f}"'] & \end{tikzcd}$$

If $R \subset \langle E\rangle$, we define

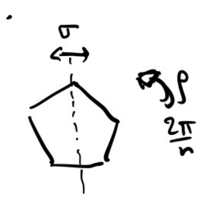
$$\langle E \mid R\rangle = \langle E\rangle / \text{normal subgroup generated by } R.$$

Universal property: $\mathrm{Hom}_{Grp}(\langle E|R\rangle, G) \cong \{ f: E \to G \mid \hat{f}(R) = \{e\} \}$.

If $E = \{e_1, \dots, e_n\}$, $R = \{r_1, \dots, r_m\}$, we also write $\langle e_1, \dots, e_n \mid r_1, \dots, r_m\rangle$.

One can also write $r = r'$ instead of $r^{-1}r'$.

Examples:
- $\langle \emptyset \rangle = \{e\}$
- $\langle a \rangle \cong \mathbb{Z}$
- $\langle a_1, \dots, a_n\rangle \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$
- $\langle a \mid a^n \rangle \cong \mathbb{Z}/n$
- $\langle a, b \mid aba^{-1}b^{-1}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$
- If G is any group, then
$$G \cong \Big\langle \{[g] \mid g\in G\} \;\Big|\; \{[gh][h]^{-1}[g]^{-1} \mid g, h \in G\} \Big\rangle$$
- D_n = dihedral group of symmetries of a regular n-gon.
$$D_n = \langle \sigma, \rho \mid \sigma^2, \rho^n, \sigma\rho\sigma\rho \rangle$$
$n=5$



- $$\begin{tikzcd} G_0 \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & G_1 \\ G_2 & \end{tikzcd}$$ with $G_i = \langle E_i \mid R_i \rangle$

Then $G_1 \underset{G_0}{*} G_2 \cong \langle E_1 \amalg E_2 \mid R_1 \amalg R_2 \amalg \{\varphi_1(g)\varphi_2(g)^{-1} \mid g \in E_0\}\rangle$

Pushouts of groupoids

Consider a pushout square of groupoids

$$\begin{tikzcd} \Gamma_0 \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Gamma_1 \arrow[d] \\ \Gamma_2 \arrow[r] & \Delta \end{tikzcd}$$

We want to describe Δ explicitly.

The functor $\mathrm{Ob}: \mathrm{Gpd} \to \mathrm{Set}$ preserves limits and colimits, because it has left and right adjoints:

$$\mathrm{disc}: \mathrm{Set} \to \mathrm{Gpd}, \quad E \mapsto \begin{cases} \mathrm{Ob}(\mathrm{disc}\, E) = E \\ \text{only identity morphisms.} \end{cases}$$

$$\mathrm{codisc}: \mathrm{Set} \to \mathrm{Gpd}, \quad E \mapsto \begin{cases} \mathrm{Ob}(\mathrm{codisc}\, E) = E \\ \mathrm{Hom}(x,y) = \{*\}, \text{ for all } x,y \in E. \end{cases}$$

disc is left adjoint to Ob: $\quad \mathrm{Hom}_{\mathrm{Gpd}}(\mathrm{disc}(E), \Gamma) \cong \mathrm{Hom}_{\mathrm{Set}}(E, \mathrm{Ob}(\Gamma))$

codisc is right adjoint to Ob: $\quad \mathrm{Hom}_{\mathrm{Gpd}}(\Gamma, \mathrm{codisc}(E)) \cong \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Ob}(\Gamma), E)$.

$$\Rightarrow \quad \mathrm{Ob}(\Delta) \cong \mathrm{Ob}(\Gamma_1) \coprod_{\mathrm{Ob}(\Gamma_0)} \mathrm{Ob}(\Gamma_2).$$

For simplicity, we assume that φ_1 and φ_2 are inclusions on objects, so

$$\mathrm{Ob}(\Delta) = \mathrm{Ob}(\Gamma_1) \cup \mathrm{Ob}(\Gamma_2).$$
$$\mathrm{Ob}(\Gamma_0) = \mathrm{Ob}(\Gamma_1) \cap \mathrm{Ob}(\Gamma_2).$$

Then:

$$\mathrm{Hom}_\Delta(x,y) = \frac{\left\{ x = x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \to \cdots \xrightarrow{f_n} x_n = y \;\middle|\; n \geq 0,\ f_i \in \mathrm{Mor}(\Gamma_1) \amalg \mathrm{Mor}(\Gamma_2) \right\}}{\text{equiv relation generated by:}}$$

- $(\cdots, f, g, \text{—}) \sim (\text{--}, h, \text{--})$ if $h = g \circ f$ in Γ_1 or Γ_2
- $(\text{--}, \mathrm{id}, \text{—}) \sim (\text{—}, \text{--})$
- $(\text{—}, f, \text{--}) \sim (\text{—}, g, \text{--})$ if $\exists\, h \in \mathrm{Mor}(\Gamma_0)$ s.t. $\varphi_1(h) = f$, $\varphi_2(h) = g$.

Easy: Δ so defined satisfies the universal property of the pushout.

Lemma The functor $B : \mathrm{Grp} \to \mathrm{Gpd}$ preserves pushouts.

Proof.

$$\begin{tikzcd} BG_0 \arrow[r] \arrow[d] & BG_1 \arrow[d] \arrow[ddr, bend left, "f"] & \\ BG_2 \arrow[r] \arrow[drr, bend right, "g"'] & B(G_1 *_{G_0} G_2) \arrow[dr, dotted, "\exists!?"] & \\ & & \Gamma \end{tikzcd} \iff \begin{tikzcd} G_0 \arrow[r] \arrow[d] & G_1 \arrow[d] \arrow[ddr, bend left] & \\ G_2 \arrow[r] \arrow[drr, bend right] & G_1 *_{G_0} G_2 \arrow[dr, dotted, "\exists!\ \checkmark"] & \\ & & \mathrm{Aut}_\Gamma(f(*)) \end{tikzcd}$$

$\square$

The following result implies the group version of the SvK theorem:

Proposition Suppose
$$\begin{tikzcd} \Gamma_0 \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Gamma_1 \arrow[d] \\ \Gamma_2 \arrow[r] & \Delta \end{tikzcd}$$
is a pushout in Gpd.

where:
- φ_1 & φ_2 are injective on objects
- Γ_0 is connected (i.e. $\exists!$ isomorphism class of objects)

Then, for any $x \in \Gamma_0$,
$$\begin{tikzcd} \mathrm{Aut}_{\Gamma_0}(x) \arrow[r] \arrow[d] & \mathrm{Aut}_{\Gamma_1}(x) \arrow[d] \\ \mathrm{Aut}_{\Gamma_2}(x) \arrow[r] & \mathrm{Aut}_\Delta(x) \end{tikzcd}$$
is a pushout in Grp.

Proof WLOG, Γ_1 & Γ_2 are connected, hence Δ is connected.

- If Γ is a connected groupoid, and $x \in \Gamma$, then $B\mathrm{Aut}_\Gamma(x) \xhookrightarrow{\simeq} \Gamma$.

  To define a quasi-inverse, we choose $\alpha_y : y \to x$ for all $y \in \Gamma$, with $\alpha_x = \mathrm{id}_x$.

  Then $\Gamma \to B\mathrm{Aut}_\Gamma(x)$ is quasi-inverse to $B\mathrm{Aut}_\Gamma(x) \hookrightarrow \Gamma$.

  $$(y_1 \xrightarrow{\gamma} y_2) \mapsto \begin{tikzcd} x \arrow[d, "\alpha_{y_1}^{-1}"'] & x \\ y_1 \arrow[r, "\gamma"'] & y_2 \arrow[u, "\alpha_{y_2}"'] \end{tikzcd}$$

  In fact, it is a <u>retraction</u>.

- Let $G_i = \mathrm{Aut}_{\Gamma_i}(x)$, $H = \mathrm{Aut}_\Delta(x)$.

  We want to show that the canonical map $G_1 *_{G_0} G_2 \to H$ is bijective.

  For each $y \in \mathrm{Ob}(\Gamma_0)$, choose $\alpha_y : y \to x$ in Γ_0 $\rightsquigarrow$ $\Gamma_0 \to BG_0$ retraction

  For each $y \in \mathrm{Ob}(\Gamma_1) \setminus \mathrm{Ob}(\Gamma_0)$, choose $\alpha_y : y \to x$ in Γ_1

  $\rightsquigarrow$ $\Gamma_1 \to BG_1$ using $\begin{cases} \alpha_y & \text{if } y \notin \Gamma_0 \\ \varphi_1(\alpha_y) & \text{if } y \in \Gamma_0. \end{cases}$

  ——— $y \in \mathrm{Ob}(\Gamma_2) \setminus \mathrm{Ob}(\Gamma_0)$ ———

  $\rightsquigarrow$ $\Gamma_2 \to BG_2$

We get functors $\Gamma_i \longrightarrow BG_i$ such that the square

$$\begin{tikzcd} \Gamma_0 \arrow[r, "\varphi_i"] \arrow[d] & \Gamma_i \arrow[d] \\ BG_0 \arrow[r] & BG_i \end{tikzcd}$$

commutes for $i=1,2$.

So we have a commutative diagram

$$\begin{tikzcd}[column sep=small] BG_0 \arrow[rr] \arrow[dr, hook] \arrow[dd] & & BG_1 \arrow[dr, hook] & & \\ & \Gamma_0 \arrow[rr] \arrow[dd] \arrow[dr] & & \Gamma_1 \arrow[dr] & \\ BG_2 \arrow[dr, hook] & & BG_0 \arrow[rr] \arrow[dd] & & BG_1 \\ & \Gamma_2 \arrow[dr] & & & \\ & & BG_2 & & \end{tikzcd}$$

where the diagonal compositions are the identity.

Taking pushouts, using the lemma, we get

$$\begin{tikzcd} B(G_1 \underset{G_0}{*} G_2) \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \Delta \arrow[r] & B(G_1 \underset{G_0}{*} G_2) \end{tikzcd}$$

$\Longrightarrow \quad G_1 \underset{G_0}{*} G_2 \longrightarrow H$ is injective.

It is also surjective:

$$H = \mathrm{Hom}_\Delta(x,x) = \left\{ x = x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \cdots \xrightarrow{f_n} x_n = x \;\middle|\; f_i \in \mathrm{Mor}(\Gamma_1) \amalg \mathrm{Mor}(\Gamma_2) \right\} \Big/ \text{relations}$$

$$\begin{tikzcd} x = x_0 \arrow[r, "f_1"] \arrow[dr] & x_1 \arrow[r, "f_2"] \arrow[d, "\alpha_{x_1}"] & x_2 \cdots \arrow[r, "f_n"] \arrow[d, "\alpha_{x_2}"] & x_n = x \\ & x \arrow[r] & x \cdots \arrow[ur] & \end{tikzcd}$$

$\Longrightarrow$ any sequence in $\mathrm{Hom}_\Delta(x,x)$ is equivalent to a sequence of automorphisms of x in either Γ_1 or Γ_2.

$\Longrightarrow \quad G_1 \underset{G_0}{*} G_2 \longrightarrow H$ is surjective. $\square$